Lecture Notes for Tensorproduct Approximation in Uncertainty Quantification taken from WS 2016/2017 by Dr. Martin Eigel and Prof. Dr. Reinhold Schneider

M. Marschall

December 8, 2016

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Part I
Gaussian Measures

1 Gaussian Measures On Hilbert Spaces

Throughout this chapter we need some notation. Let us denote a general *probability space* by $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the basis set which will be equipped with a *σ-algebra* $\mathcal{F}$. $\mathbb{P}$ is a *probability measure*. For every $A \in \mathcal{F}$, A^c denotes the *complement* of A. For every complete metric space we can define a measurable space by $(E, \mathcal{B}(E))$, where the involved σ-algebra is the *Borel-σ-algebra*, which is the collection of all open/closed subsets of E. A *random variable* (RV) X, in $(\Omega, \mathcal{F}, \mathbb{P})$ with values in E, is a measurable mapping $X : \Omega \to E$, i.e.

$$I \in \mathcal{B}(E) \Rightarrow X^{-1}(I) \in \mathcal{F} \ . \tag{1.1}$$

The *law* of X is a probability measure $\mathbb{P}_X$ on $(E, \mathcal{B}(E))$, s.t for all $I \in \mathcal{B}$ we have

$$\mathbb{P}_X(I) := \mathbb{P}(X^{-1}(I)) =: \mathbb{P}(X \in I) \tag{1.2}$$

One of the important classical results is the *change of variables*, which is subject to a bounded mapping $\varphi : E \to \mathbb{R}$ and describes the property

$$\int_\Omega \varphi(X(\omega))\mathbb{P}(d\omega) = \int_E \varphi(x)\mathbb{P}_X(dx) \tag{1.3}$$ 2

By H we will denote a real separable Hilbertspace with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. Furthermore, $L(H)$ is the *Banach algebra of continuous linear operators from H into H*. Whereas, $L^+(H)$ is the set of all symmetric and positive $T \in L(H)$, i.e.,

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \qquad \langle Tx, x \rangle \geq 0 \quad \text{for } x, y \in H \tag{1.4}$$

Moreover, $L_1^+(H)$ is the set of trace class operators $Q \in L^+(H)$, i.e

$$\mathrm{Tr}(Q) := \sum_{k=0}^{\infty} \langle Qe_k, e_k \rangle < \infty \tag{1.5}$$

for one (and hence for all) ONS $(e_k)_{k=1}^\infty$

Remark 1.1. If $Q \in L_1^+(H)$, then Q is compact and $\mathrm{Tr}(Q)$ is the sum of its eigenvalues (repeated according to its multiplicity).

1.1 One-Dimensional Hilbert Spaces

For $a \in \mathbb{R}$ and $\lambda \geq 0$ define a probability measure $\mathcal{N}_{a,\lambda}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by either

$$\lambda = 0 \qquad \mathcal{N}_{a,0} = \delta_a \tag{1.6}$$

with Dirac measure

$$\delta_a(B) = \begin{cases} 1 & , \quad \text{if } a \in B \\ 0 & , \quad \text{else.} \end{cases} \tag{1.7}$$

or, for $\lambda > 0$, by

$$\mathcal{N}_{a,\lambda}(B) = \frac{1}{\sqrt{2\pi\lambda}} \int_B e^{-\frac{(x-a)^2}{2\lambda}} dx \ , \tag{1.8}$$

for any $B \in \mathcal{B}(\mathbb{R})$.

Remark 1.2. $\mathcal{N}_{a,\lambda}$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, since $\mathcal{N}_{a,\lambda}(\mathbb{R}) = 1$.

Proposition 1.3. *For $\lambda > 0$, $\mathcal{N}_{a,\lambda}$ is absolutely continuous w.r.t the Lebesgue measure.*

To see the correspondence to Gaussian measures, we have a look at some properties.

Theorem 1.4. *Let $a \in \mathbb{R}, \lambda > 0$, then the following identities hold*

$$\int_{\mathbb{R}} x \mathcal{N}_{a,\lambda}(dx) = a \ , \tag{1.9}$$

$$\int_{\mathbb{R}} (x-a)^2 \mathcal{N}_{a,\lambda}(dx) = \lambda \ , \tag{1.10}$$

for $h \in \mathbb{R}$

$$\hat{\mathcal{N}}_{a,\lambda}(h) = \int_{\mathbb{R}} e^{ihx} \mathcal{N}_{a,\lambda}(dx) = e^{iah - \frac{1}{2}\lambda h^2} \tag{1.11}$$

We call a the mean, *λ the* variance *and $\hat{\mathcal{N}_{a,\lambda}}$ the* characteristic function.

Proof. Exercise. ☐

1.2 Finite-Dimensional Hilbert Spaces

Assume H is a d-dimensional space with $d < \infty, d \in \mathbb{N}$. In the following we will construct the concept of *product probabilities*.

Definition 1.5. Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i = 1, \dots, d$ be probability spaces. Then, the *product space* is given by $\Omega = \Omega_1 \times \dots \times \Omega_d$. A measurable rectangle $R \subset \Omega$ has the form $R = B_1 \times \dots \times B_d$, $B_i \in \mathcal{F}_i$, $i = 1, \dots, d$, and the $\sigma-$algebra $\mathcal{F}$, generated by the set of all measurable rectangles, is called *product σ-algebra* of $\mathcal{F}_i$, $i = 1, \dots, d$. For any measurable rectangle R set $\tilde{\mathbb{P}}(R) := \mathbb{P}_1(B_1) \cdot \dots \cdot \mathbb{P}_d(B_d)$. Then, $\tilde{\mathbb{P}}$ can be uniquely extended to a probability measure on $(\Omega, \mathcal{F})$, denoted by $\mathbb{P} := \mathbb{P}_1 \otimes \dots \otimes \mathbb{P}_d$.

To define Gaussian measures on finite dimensional Hilbert spaces we will have a look at the same concepts as in the one-dimensional case. Let $Q \in L^+(H)$, $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and $(e_k)_{k=1}^d$ an orthonormal basis on H, s.t. there exists $(\lambda_k)_{k=1}^d$,$\lambda_k \geq 0$ with $Qe_k = \lambda_k e_k, k = 1, \dots, d$. Furthermore, for $x \in H$, we set

$$x_k = \langle x, e_k \rangle, \quad k = 1, \dots, d \ . \tag{1.12}$$

Proposition 1.6. *Any real finite dimensional Hilbert space H can be identified with $\mathbb{R}^d$ by an isomorphism*

$$\gamma\colon H \to \mathbb{R}^d, \quad x \mapsto \gamma(x) = (x_1, \dots, x_d). \tag{1.13}$$

For $a \in \mathbb{R}^d$ and $Q \in L^+(H)$ define a probability measure $\mathcal{N}_{a,Q}$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by

$$\mathcal{N}_{a,Q} = \bigotimes_{k=1}^{d} \mathcal{N}_{a_k,\lambda_k} \tag{1.14}$$

Now, as in the one-dimensional case, we have a look at the mean, the *covariance*, and the characteristic function.

Theorem 1.7. *Let $a \in H$, $Q \in L^+(H)$ and $\mu = \mathcal{N}_{a,Q}$, then the following identites hold*

$$\int_H x\mathcal{N}_{a,Q}d(x) = a, \tag{1.15}$$

for $y, z \in H$

$$\int_H \langle y, x-a\rangle\langle z, x-a\rangle \mathcal{N}_{a,Q}(dx) = \langle Qy, z\rangle, \tag{1.16}$$

for $h \in H$

$$\hat{\mathcal{N}}_{a,Q}(h) := \int_H e^{i\langle x,h\rangle}\mathcal{N}_{a,Q}(dx) = e^{i\langle a,h\rangle - \frac{1}{2}\langle Qh,h\rangle}. \tag{1.17}$$

We call a the mean*, Q the* covariance *and $\widehat{\mathcal{N}_{a,Q}}$ the* characteristic function*. For $det(Q) > 0$, $\mathcal{N}_{a,Q}$ is absolutely continuous w.r.t the Lebesgue measure in $\mathbb{R}^d$ and*

$$\mathcal{N}_{a,Q}(dx) = \frac{1}{\sqrt{(2\pi)^d det(Q)}} e^{-\frac{1}{2}\langle Q^{-1}(x-a), x-a\rangle} dx \tag{1.18}$$

is the corresponding Lebesgue density.

Proof. Exercise. □

1.3 General Hilbert Spaces

Let H be an infinite dimensional Hilbert space and $(e_k)_{k=1}^{\infty}$ a complete orthonormal system in H. For $n \in \mathbb{N}$, define the finite projection $P_n : H \to P_n(H)$

$$P_n x = \sum_{k=1}^{n} \langle x, e_k\rangle e_k, \qquad x \in H \tag{1.19}$$ eq:finiteProjection

with $\lim_{n\to\infty} \mathbb{P}_n x = x$ for $x \in H$.

First of all, let us have a look at the existence of Gaussian measures.

Definition 1.8. A measure μ on $(H, \mathcal{B}(H))$ is called *Gaussian measure*, if for any $x \in H$ the random variable $\langle x, \cdot\rangle$ has a Gaussian distribution. [see later in Theorem 1.16].

Thm:Exist-mu **Theorem 1.9.** *For any probability measure μ on $(H, \mathcal{B}(H))$ with finite first and second moment there exists unique $m \in H$ and $Q \in L(H)$, s.t. it holds, for $h \in H$*

$$\langle m, h\rangle = \int_H \langle x, h\rangle \mu(dx) \tag{1.20}$$

and for $h, k \in H$

$$\langle Qh, k\rangle = \int_H \langle h, x-m\rangle\langle k, x-m\rangle \mu(dx)\ . \tag{1.21}$$

Again, we call m the mean *and Q the* covariance *of μ.*

To proof this claim, we need some basic properties.

lemma:Exist-m **Lemma 1.10.** *Let μ be a probability measure on $(H, \mathcal{B}(H))$ with finite first moment, i.e.*

$$\int_H |x|\mu(dx) < \infty. \tag{1.22}$$ eq:finitefirstmoment

Then, the linear functional

$$F \colon H \to \mathbb{R},\ F(h) = \int_H \langle x, h\rangle \mu(dx) \quad \text{for } h \in H \tag{1.23}$$

is continuous.

Proof. This can be seen by

$$|F(h)| \leq \int_H |x|\mu(dx)|h|. \tag{1.24}$$

□

lemma:Exist-Q **Lemma 1.11.** *Let μ be a probability measure on $(H, \mathcal{B}(H))$ with finite second moment, i.e.*

$$\int_H |x|^2\mu(dx) < \infty. \tag{1.25}$$ eq:finitesecondmoment

Then, the bilinear form $G \colon H \times H \to \mathbb{R}$, for $h, k \in H$, given by

$$G(h, k) = \int_H \langle h, x - m\rangle\langle k, x - m\rangle \mu(dx) \tag{1.26}$$

is continuous

Proof. We can bound G by

$$|G(h, k) \leq \int_H |x - m|^2\mu(dx)|h||k|. \tag{1.27}$$

□

Proof of Theorem 1.9. By 1.10, 1.11 and Riesz representation theorem there exists unique $m \in H$ and a linear and bounded operator $Q \in L(H)$, with the above properties. □

We write

$$m = \int_H x\mu(dx)\ . \tag{1.28}$$

Remark 1.12. Note that $Q \in L_1^+(H)$, i.e. it is symmetric, positive and of trace class.

Obviously, this can be done the other way round.

Theorem 1.13. *Let there be a mean value $a \in H$ and a covariance operator $Q \in L_1^+(H)$. Then, the Gaussian measure $\mu = \mathcal{N}_{a,Q}$ on $(H, \mathcal{B}(E))$ is the unique measure with mean a, covariance Q and Fourier transformation*

$$\hat{\mathcal{N}}_{a,Q}(h) := exp\{i\langle a, h\rangle - \frac{1}{2}\langle Qh, h\rangle\} \quad , h \in H\ . \tag{1.29}$$

We call $\mathcal{N}_{a,Q}$ nondegenerated, *if $ker(Q) = \{x \in H | Qx = 0\} = \{0\}$.*

Proof. Exercise (use 1.19). □

Proposition 1.14. *It holds, $\mu = \bigotimes_{k=1}^{\infty} \mathcal{N}_{a_k,\lambda_k}$, where $Qe_k = \lambda_k e_k$ for the complete orthonormal system $(e_k)_{k=1}^{\infty}$.*

Remark 1.15. Due to the isomorphism γ between H and l^2, μ is a Gaussian measure on $H = l^2$. Furthermore, l^2 is a Borel subset of $\mathbb{R}^\infty$ and μ is concentrated in l^2, i.e.

$$\mu(l^2) = 1 \ . \tag{1.30}$$

1.4 Gaussian random variables (RVs)

Let K a Hilbert space and X a K-valued Gaussian RV in $(\Omega, \mathcal{F}, \mathbb{P})$. For $p \geq 1$, $L^p(\Omega, \mathcal{F}, \mathbb{P}; K)$ denotes the space of p-integrable RVs $X\colon \Omega \to K$, i.e.

$$\int_\Omega |X(\omega)|^p \mathbb{P}(d\omega) < \infty \ . \tag{1.31}$$

The space $L^p(H, \mathcal{B}(H), \mu; K)$ embedded with the norm

$$\|X\|_{L^p(\Omega,\mathcal{F},\mathbb{P};K)} = \left(\int_\Omega |X(\omega)|^p \mu(d\omega) \right)^{\frac{1}{p}} \tag{1.32}$$

is a Banach space. Now, we want to show that, under certain assumptions, for every K-valued random variable there exists a Gaussian measure on K.

Thm: RV-Gaussian exists

Theorem 1.16. *Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; K)$. Then, there exists unique $m_X \in K$ and $Q \in L^+(K)$, s.t. for $h, k \in K$ we have*

$$\langle m_x, h \rangle = \int_K \langle y, h \rangle \mathbb{P}_X(dy) = \int_\Omega \langle X(\omega), h \rangle \mathbb{P}(d\omega) \ , \tag{1.33}$$

$$\langle Qh, k \rangle = \cdots = \int_\Omega \langle X(\omega) - m_X, h \rangle \langle X(\omega) - m_X, k \rangle \mathbb{P}(d\omega) \tag{1.34}$$

for given Fourier transform

$$\hat{\mu_X}(k) = \int_K e^{i\langle y,k \rangle} \mathbb{P}_X(dy) = \int_\Omega e^{i\langle X(\omega),k \rangle} \mathbb{P}(d\omega). \tag{1.35}$$

Proof. Use 1.3 and Theorem 1.9. □

Example 1.17. *Assume $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ is a real valued Gaussian random variable with law $\mathcal{N}_\lambda$. Then, for $m \in \mathbb{N}$,*

$$\int_\Omega |X(\omega)|^{2m} \mathbb{P}(d\omega) = (2\pi\lambda)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \xi^{2m} e^{-\frac{\xi^2}{2}} d\xi = \frac{(2m!)}{2^m m!} \lambda^m \ . \tag{1.36}$$

Hence , $X \in L^{2m}(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for $m \in \mathbb{N}$.

Definition 1.18. Let $X_1, \ldots, X_n$ be real RVs in H and consider X as an $\mathbb{R}^n$-valued random variable

$$X(\omega) = (X_1(\omega), \ldots, X_n(\omega)) \ . \tag{1.37}$$

Then, $X_1, \ldots, X_n$ are *independent* if

$$\mathbb{P}_X = \bigotimes_{i=1}^{n} \mathbb{P}_{X_i} \ . \tag{1.38}$$

Proposition 1.19. *The real valued random variables* $X_1, \dots X_n$ *are independent if and only if for any set of real positive Borel functions* $\varphi_1, \dots, \varphi_n$ *it holds*

$$\int_\Omega \prod_{k=1}^n \varphi_k(X_k(\omega_k))\mathbb{P}(d(\omega_1, \dots, \omega_n)) = \prod_{k=1}^n \int_{\Omega_k} \varphi_k(X_k(\omega))\mathbb{P}_k(d\omega_k) \qquad (1.39)$$ 3

Example 1.20. *Let* $\Omega = H$, $\mu = \mathcal{N}_Q := \mathcal{N}_{0,Q}$ *and* $(e_k)_{k=1}^\infty$ *an orthonormal basis in* H *and* $(\lambda_k)_{k=1}^\infty$ *positive numbers, s.t*

$$Qe_k = \lambda_k e_k \ , \quad k \in \mathbb{N} \ . \qquad (1.40)$$

Moreover, define the RVs $X_i(x) = \langle x, e_i \rangle$, *for* $i = 1, \dots, n$ *and* $x \in H$. *Then,*

$$\mu_{X_i} = \mathcal{N}_{\langle Qe_i, e_i \rangle} = \mathcal{N}_{\lambda_i} := \mathcal{N}_{0,\lambda_i} \qquad (1.41)$$

and

$$\mu_X = \mathcal{N}_{Q_{i,j}} \ , \qquad (1.42)$$

where $Q_{i,j} := \langle Qe_i, e_j \rangle = \lambda_i \delta_{i,j}$, $i, j = 1, \dots, n$. *Hence,*

$$\mu_X = \bigotimes_{i=1}^n \mathcal{N}_{\lambda_i} = \bigotimes_{i=1}^n \mu_{X_i} \qquad (1.43)$$

and, the $X_1, \dots, X_n$ *are independent.*

Example 1.21. *Let* $H = \mathbb{R}^n$, $Q = (Q_{i,j}) \in L^+(\mathbb{R}^n)$, *s.t.* $det(Q) > 0$ *and* $\mu = \mathcal{N}_Q$, *s.t.*

$$\mu(dx) = \frac{1}{\sqrt{(2\pi)^n det(Q)}} e^{-\frac{1}{2}\langle Q^{-1}x, x \rangle} dx, \quad x \in \mathbb{R}^n \qquad (1.44)$$

Furthermore, let μ *be the law of an* $\mathbb{R}^n$ *valued RV* X. *Set* $X_i(\omega) = (X(\omega))_i$ *for* $i = 1, \dots, n$. *Then, it holds*

$$\mu_{X_j} = \mathcal{N}_{Q_{j,j}}, \quad j = 1, \dots, n \qquad (1.45)$$

and

$$\mu_X = \mathcal{N}_{(Q_{i,j})} \ . \qquad (1.46)$$

Hence, the RVs $X_1, \dots, X_n$ *are independent if and only if* Q *is diagonal.*

Example 1.22. *Let, for* $Q \in L_1^+(H)$, $\mu = \mathcal{N}_Q$ *be a Gaussian measure on* $(H, \mathcal{B}(H))$ *and* $z_i \in H$ *for* $i = 1, \dots, n$. *Set, for* $x \in H$ *and* $i = 1, \dots n$

$$X_{z_i}(x) = \langle x, z_i \rangle \qquad (1.47)$$

and $X = (X_{z_1}, \dots X_{z_n})$. *Then,* $X_{z_1}, \dots X_{z_n}$ *are independent if and only if for* $i, j = 1, \dots, n$

$$\langle Qz_i, z_j \rangle = 0, \quad \text{if } z_i \neq z_j. \qquad (1.48)$$

1.5 Cameron-Martin Space And White Noise Mapping

Assume a separable infinite dimensional Hilbert space H and a nondegenerated Gaussian measure $\mu = \mathcal{N}_Q$. i.e. $Q \in L_1^+(H)$ and Ker $Q = \{0\}$. Note that Q^{-1} is not continuous, since for any CONS $(e_k)_{k=1}^{\infty}$

$$Q^{-1} e_k = \frac{1}{\lambda_k} e_k \tag{1.49}$$

for $k \in \mathbb{N}$ and $\lambda_k \to \infty$ as $k \to \infty$.

Lemma 1.23. *Assume H is an infinite dimensional Hilbert space with CONS $(e_k)_{k=1}^{\infty}$. Then, $Q(H)$ is a dense subspace of H and given by*

$$Q(H) = \left\{ x \in H \colon \sum_{k=1}^{\infty} \langle x, e_k \rangle^2 \lambda_k^{-2} < \infty \right\} \tag{1.50}$$

Proof e.q. Da Prato: Functional Analytic Methods for Ev. Eq. Lemma 2.8. The identity follows directly and $Q(H)$ beeing dense in H inherits from the trivial kernel of Q. □

In the following, it will be usefull to define $Q^{1/2}$. For any $x \in H$ we have

$$Q^{1/2} x = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle x, e_k \rangle e_k \ . \tag{1.51}$$

The space $Q^{1/2}$ is called a *Cameron-Martin* space. In the theory of stochastic partial differential equations, this space will also be called the *reproducing kernel* of μ. It is easy to see that, $Q^{1/2}(H)$ is a proper subspace of H, since

$$Q^{1/2}(H) = \left\{ x \in H \colon \sum_{k=1}^{\infty} \langle x, e_k \rangle^2 \lambda_k^{-1} < \infty \right\} \ . \tag{1.52}$$

Given $z \in Q^{1/2}(H)$ we consider the function $W_z \in L^2(H, \mu) := L^2(H, \mathcal{B}(H), \mu)$ defined for any $x \in H$ by

$$W_z(x) := \langle Q^{-1/2} z, x \rangle \ . \tag{1.53}$$

It will be important to define this for every $z \in H \supset Q^{1/2}(H)$. Intuitively setting

$$W_z(x) = \langle Q^{-1/2} z, x \rangle \tag{1.54}$$

for $x \in H$ is missleading, since one can show that

$$\mu(Q^{1/2}(H)) = 0 \ . \tag{1.55}$$

Alternatively, for $x \in H$, we consider the mapping

$$W \colon Q^{1/2}(H) \subset H \to L^2(H\mu),\ z \mapsto W_z,\ W_z(x) = \langle x, Q^{-1/2} z \rangle \tag{1.56}$$

W is an isometry because, for all $z_1, z_2 \in Q^{1/2}(H)$,

$$\int_H W_{z_1}(x) W_{z_2}(x) \mu(dx) = \langle Q Q^{-1/2} z_1, Q^{-1/2} z_2 \rangle = \langle z_1, z_2 \rangle \ . \tag{1.57}$$

Since, $Q^{\frac{1}{2}}(H)$ is dense in H, the *white noise mapping* W can be uniquely extended to H.

Proposition 1.24. *Let $z_1, \dots, z_n \in H$. Then, the law of the $\mathbb{R}^n$-valued RV $W = (W_{z_1}, \dots, W_{z_n})$ is given by*

$$\mu_W = \mathcal{N}_{(\langle z_i, z_j \rangle)_{i,j=1,\dots,n}} \tag{1.58}$$ 4

Moreover, the RVs $W_{z_1}, \dots, W_{z_n}$ are independent if and only if $(z_1, \dots z_n)$ are orthogonal, i.e.

$$\langle z_i, z_j \rangle = \delta_{i,j} \ , \quad i,j = 1, \dots, n \ . \tag{1.59}$$

Exercise 1.25. *For $z \in H$ we have*

$$\int_H e^{W_z}(x)\mu(dx) = e^{\frac{1}{2}|z|^2} \tag{1.60}$$

Exercise 1.26. *The function $H \to L^2(H,\mu)$, $f \mapsto e^{W_f}$ is continuous.*

Exercise 1.27. *For $f, g \in H$ we have*

$$\int_H W_f W_g d\mu = \langle f, g \rangle \ . \tag{1.61}$$

2 L^2 Spaces With Respect To Gaussian Measures

Denote by $L^2(H, \mathcal{B}(H))$ the Hilbert space of all equivalence classes of Borel square integrable real functions on H embedded with the inner product

$$\langle \varphi, \psi \rangle_{L^2(H,\mu)} = \int_H \varphi \psi d\mu \ , \quad \varphi, \psi \in L^2(H,\mu) \tag{2.1}$$

and the norm

$$\|\varphi\|_{L^2(H,\mu)} = \left(\int_H |\varphi|^2 d\mu \right)^{\frac{1}{2}} \ . \tag{2.2}$$

Denote by $L^2(H,\mu;H)$ the Hilbert space of all equivalence classes of Borel square integrable mappings from H to H. This space will be called L^2 vector field. For $F\colon H \to H$ define

$$\|F\|_{L^2(H,\mu;H)} := \left(\int_H |F(x)|^2 \mu(dx) \right)^{\frac{1}{2}} < \infty \tag{2.3}$$

and for another $G\colon H \to H$

$$\langle F, G \rangle_{L^2(H,\mu;H)} := \int_H \langle F(x), G(x) \rangle \mu(dx). \tag{2.4}$$

2.1 Orthonormal basis in $L^2(H,\mu)$

2.1.1 One-dimensional case $H = \mathbb{R}, \mu = \mathcal{N}_1$

We define an orthonormal basis in terms of *Hermite* polynomials. Therefore, consider for $t, \xi \in \mathbb{R}$ the analytic functions

$$F(t,\xi) = e^{-\frac{t^2}{2} + t\xi} \tag{2.5}$$

and the polynomials $(H_n)_{n\in\mathbb{N}_0}$, given by

$$F(t,\xi) = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(\xi) \tag{2.6}$$

where for $n \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$ it holds

$$H_n(\xi) = \frac{(-1)^n}{\sqrt{n!}} e^{\frac{\xi^2}{2}} D_\xi^n (e^{-\frac{\xi^2}{2}}) \ . \tag{2.7}$$

The Hermite polynomial H_n has degree n and a positive leading coefficient.

Example 2.1. *The first Hermite polynomials are*

$$H_0(\xi) = 1 \tag{2.8}$$

$$H_1(\xi) = \xi \tag{2.9}$$

$$H_2(\xi) = \frac{1}{\sqrt{2}}(\xi^2 - 1) \ . \tag{2.10}$$

Furthermore, it holds a recursive structure

$$\xi H_n(\xi) = \sqrt{n+1} H_{n+1}(\xi) + \sqrt{n} H_{n-1}(\xi) \tag{2.11}$$

$$D_\xi H_n(\xi) = \sqrt{n} H_{n-1}(\xi) \tag{2.12}$$

$$D_\xi^2 H_n(\xi) = \xi D_\xi H_n(\xi) = -n H_n(\xi) \ . \tag{2.13}$$

The system $(H_n)_{n\in\mathbb{N}_0}$ is orthonormal and complete in $L^2(\mathbb{R},\mu)$.

2.2 The Infinite dimensional case

Our aim is to construct a complete orthonormal system on $L^2(H,\mu)$ in terms of *generalized Hermite polynomials.*

Definition 2.2. Let $\gamma\colon \mathbb{N} \to \mathbb{N}_0$, $n \mapsto \gamma_n$, s.t.

$$|\gamma| := \sum_{k=1}^{\infty} \gamma_k < \infty \tag{2.14}$$

γ can be interpreted as finite multi-index. The set of all mappings of that kind is called $\mathit{\Gamma}$. One can see, if $\gamma \in \mathit{\Gamma}$, then $\gamma_n = 0$ for almost all $n \in \mathbb{N}$. This defines the (tensorized) *Hermite polynomials.*

$$H_\gamma(x) = \prod_{k=1}^{\infty} H_{\gamma_k}(W_{e_k}(x)), \quad x \in H, \tag{2.15}$$

where W_z is the white noise mapping

$$W\colon H \to L^2(H,\mu),\ z \mapsto W_z,\ W_z(x) = \langle x, Q^{-\frac{1}{2}} z\rangle, \text{ for } x \in H. \tag{2.16}$$

Note that almost all factors (with the exception of finitely many terms) are $H_0(W_{e_k}(x)) = 1$, $x \in H$.

Exercise 2.3. *The system $(H_\gamma)_{\gamma\in\mathit{\Gamma}}$ is orthonormal and complete on $L^2(H,\mu)$.*

2.2.1 Wiener-Itô decomposition

For all $n \in \mathbb{N}$, denote $L^2_n(H,\mu)$ the closed subspace of $L^2(H,\mu)$ spanned by

$$\{H_n(W_f) \colon f \in H, |f| = 1\} \tag{2.17}$$

In particular $L^2_0(H,\mu)$ is the space of all constants and $L^2_1(H,\mu)$ is the space of all Gaussian random variables, which belongs to H, i.e.

$$L^2_1(H,\mu) = \operatorname{span}\{W_f \colon f \in H\}. \tag{2.18}$$

Denote by Π_n the orthogonal projector of $L^2(H,\mu)$ onto $L^2_n(H,\mu)$, $n \in \mathbb{N}_0$.

Lemma 2.4. *It holds*

$$L^2(H,\mu) = \bigoplus_{n=0}^{\infty} \Pi_n L^2(H,\mu) = \bigoplus_{n=0}^{\infty} L^2_n(H,\mu). \tag{2.19}$$

This decomposition is called *Wiener-Itô* (or *chaos*) decomposition of $L^2(H,\mu)$. $L^2_n(H,\mu)$ are the components of the decomposition.

Lemma 2.5 (characterization of $L^2_n(H,\mu)$)**.**
For $n \in \mathbb{N}$, $L^2_n(H,\mu)$ coincides with the closed subspace of $L^2(H,\mu)$ spanned by

$$V_n := \{H_\gamma \colon |\gamma| = n\} \tag{2.20}$$

Proposition 2.6. *Let $p(\xi)$ be a real polynomial of degree $n \in \mathbb{N}$ and $f \in H$ with $|f| = 1$. It follows*

$$p(W_f) \in \bigoplus_{k=1}^{n} L^2_k(H,\mu) \tag{2.21}$$

and

$$p(W_f) = \sum_{k=0}^{n} c_k H_k(W_f) \tag{2.22}$$

for some $c_1, \ldots, c_n \in \mathbb{R}$.

Part II
Galerkin methods for random PDEs

3 Hermite and generalized polynomial chaos

We assume H is an infinite dimensional Hilbert space over $\mathbb{R}$ and $\mu = \mathcal{N}_Q$ nondegenerated, centered Gaussian with $Qe_m = \lambda_m e_n$, $m \in \mathbb{N}$. Furthermore, let $(Y_m)_{m\in\mathbb{N}}$ be a sequence of RVs on (H,μ) with

$$Y_m(x) := W_{e_m}(x) = \langle x, Q^{-\frac{1}{2}} e_m\rangle = \lambda_m^{-\frac{1}{2}} \langle x, e_m\rangle, \quad m \in \mathbb{N}. \tag{3.1}$$

$(Y_m)_{m\in\mathbb{N}}$ are independent and identically distributed $Y_1 \sim \mathcal{N}_1$. The distribution of the injective map

$$Y\colon H \to \mathbb{R}^\infty,\, x \mapsto (Y_m(x))_{m\in\mathbb{N}} \tag{3.2}$$

is the countable product measure

$$\gamma := \mu_Y = \bigotimes_{m\in\mathbb{N}} \mathcal{N}_1 \tag{3.3}$$

on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^\infty)$.

Proposition 3.1. *The pullback*

$$Y^*\colon L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \gamma) \to L^2(H,\mu),\; f \mapsto Y^* f = f \circ Y \tag{3.4}$$ 5

is an isometric isomorphism.

Proof. Since γ is the image measure of Y under μ, we have, for all $f \in L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \gamma)$

$$\|Y^* f\|^2_{L^2(H,\mu)} = \|f\|^2_{L^2(\mathbb{R}^\infty,\mathcal{B}(\mathbb{R}^\infty),\gamma)}, \tag{3.5}$$

by the transformation theorem. Hence, Y^* is an isometry. By H being separable and the topology of H is generated by Y, also the Borel σ-algebra is generated by Y. Therefore, the Doob-Dynkin lemma implies, for every $f \in L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \gamma)$, the existence of a measurable function $X \in L^2(H,\mu)$, with $\sigma(X) \subset \sigma(Y)$ and $X = f \circ Y$. Hence, Y^* is surjective. ☐

Define the index set of finitely supported sequences in $\mathbb{N}$ by

$$\mathcal{F} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} \colon |\text{supp } \nu| < \infty\} \tag{3.6}$$

where

$$\text{supp } \nu := \{m \in \mathbb{N}, \nu_m \neq 0\}. \tag{3.7}$$

For $y \in \mathbb{R}^\infty$ and $\nu \in \mathcal{F}$ define the tensorized Hermite polynomials by

$$H_\nu(y) = \prod_{m\geq 1} H_{\mu_m}(y_m) = \prod_{m\in \text{supp } \nu} H_{\nu_m}(y_m) \tag{3.8}$$ tenHermite

Note that 3.8 is a polynomial of degree $|\nu| := \sum_{m\in\text{supp } \nu} \nu_m$. Using the pullback operator and 1.39, we have that, for every $x \in H$

$$H_\nu(x) := (Y^* H_\nu)(x) = H_\nu(\gamma(x)) = \prod_{m\in\mathbb{N}} H_{\nu_m}(W_{e_m}(x)) \tag{3.9}$$

is a polynomial on H.

Exercise 3.2. $(H_\nu)_{\nu\in\mathcal{F}}$ *is orthonormal basis of* $L^2(H,\mu)$.

Hint. Use the isometric isomorphism Y^*. ☐

3.1 Generalized Polynomial Chaos

An orthonormal basis may be constructed if the probability space (H, μ) exhibits a countable product structure, illustrated by the measure-preserving map Y into the product measure space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \gamma)$. For all $m \in \mathbb{N}$, Γ_m be an arbitrary non-empty set endured with the σ-algebra Σ_m. Let $(Y_m)_{m\in\mathbb{N}}$ be independent RVs on the probability space $(\Omega, \Sigma_\Omega, \mathbb{P})$, s.t. Y_m maps into (Γ_m, Σ_m). This constitutes a map

$$Y \colon \Omega \to \Gamma := \bigtimes_{m\in\mathbb{N}} \Gamma_m,\ \omega \mapsto (Y_m(\omega))_{m\in\mathbb{N}} \tag{3.10}$$ 6

which is a measurable map w.r.t the product σ-algebra $\Sigma := \bigotimes_{m\in\mathbb{N}} \Sigma_m$ on Γ. By independence of $(Y_m)_{m\in\mathbb{N}}$, the distribution of Y is the countable product probability measure

$$\mu := \bigotimes_{m\in\mathbb{N}} \mu_m := \bigotimes_{m\in\mathbb{N}} \mathbb{P}_{Y_m} \tag{3.11}$$

on (Γ, Σ). For all $m \in \mathbb{N}$ let $(\varphi_{m,i})_{i\in\mathbb{N}_0}$ be an orthonormal basis of the space $L^2(\Gamma_m, \Sigma_m, \mu_m)$, s.t. $\varphi_{m,0} \equiv 1$ (normalized, since μ_m is a probability measure). For $\nu \in \mathcal{F}$ define the tensor product

$$\varphi_\nu := \bigotimes_{m\in\mathbb{N}} \varphi_{m,\nu_m} \tag{3.12}$$

and let $y = (y_m)_{m\in\mathbb{N}} \in \Gamma$. Note that y has to be interpreted as a realisation of the random variable Y for some $\omega \in \Omega$. With $F(\mathbb{N})$ define the set of all finite subsets of $\mathbb{N}$ and $I \in F(\mathbb{N})$. Furthermore, define the finite product σ-algebra

$$\Sigma_I := \bigotimes_{m\in I} \Sigma_m \subset \Sigma. \tag{3.13}$$

A function is Σ_I-measurable if it is Σ-measurable and only depends on $(y_m)_{m\in I}$. Set

$$\mathcal{F}_I := \{\nu \in \mathcal{F} \colon \operatorname{supp} \nu \subset I\}. \tag{3.14}$$

Lemma 3.3. *For all $I \in F(\mathbb{N})$, the $(\varphi_\nu)_{\nu\in\mathcal{F}_I}$ form an orthonormal basis of $L^2(\Gamma, \Sigma_I, \mu)$.*

Proof. Use

$$L^2(\Gamma, \Sigma_I, \mu) \cong \bigotimes_{m\in I} L^2(\Gamma_m, \Sigma_m, \mu_m) \tag{3.15}$$

and I being finite. □

Proposition 3.4. *The set*

$$\bigcup_{I\in F(\mathbb{N})} L^2(\Gamma, \Sigma_I, \mu) \tag{3.16}$$

is dense in $L^2(\Gamma, \Sigma, \mu)$.

Proof. Application of the monotone class theorem. □

Finally the main result.

Theorem 3.5. *$(\varphi_\nu)_{\in\mathcal{F}}$ is an orthonormal basis of $L^2(\Gamma, \Sigma, \mu)$.*

Proof. Exercise! □

3.2 Orthogonal Polynomials

Assume Γ_m is a Borel subset of $\mathbb{R}$ and μ_m has finite moments

$$M_n := \int_{\Gamma_m} \xi^n \mu_m(d\xi), \quad n \in \mathbb{N}_0. \tag{3.17}$$ 7

Then, the orthogonal polynomials w.r.t μ_m can be constructed by a three-term recursion

$$\beta_{n+1} P_{n+1}(\xi) = (\xi - \alpha_n) P_n(\xi) - \beta_n P_{n-1}(\xi), \quad n \in \mathbb{N}_0, \tag{3.18}$$ 8

with $P_{-1}(\xi) = 0$, $P_0(\xi) = 1$ and

$$\alpha_n := \int_{\Gamma_m} \xi P_n(\xi) \mu_m(d\xi) \tag{3.19}$$

$$\beta_n := \frac{c_n - 1}{c_n} \tag{3.20}$$

and $\beta_0 = 1$, where c_n is the leading coefficient of P_n. 3.18 can be divided by Gram-Schmidt orthogonalisation of monomials $(\xi^n)_{n \in \mathbb{N}_0}$. For all $n \in \mathbb{N}_0$, P_n is a polynomial of degree n if $n < N := \dim(L^2(\Gamma_m, \Sigma_m, \mu_m))$ and zero otherwise. The sequence of $(P_n)_{n \in \mathbb{N}_0}$ (resp. $(P_n)_{n=0}^{N-1}$, if N is finite) is orthonormal in $L^2(\Gamma_m, \Sigma_m, \mu_m)$. For N finite, it follows immediately that $(P_n)_{n=0}^{N}$ is an orthonormal basis of $L^2(\Gamma_m, \Sigma_m, \mu_m)$. With $N = \infty$, for this to hold, μ_m has to be *determinate*, i.e. the measure μ_m is uniquely characterized by its moments $(M_n)_{n \in \mathbb{N}_0} \in \mathbb{R}$. Note that for bounded $\Gamma_m \subset \mathbb{R}$, μ_m is determinate.

Example 3.6. *For $m \in \mathbb{N}_0$ let $(P_n^m)_{n \in \mathbb{N}_0}$ be orthonormal polynomials forming a basis of $L^2(\Gamma_m, \Sigma_m, \mu_m)$. Then, the tensor product polynomials*

$$P_\nu := \bigotimes_{m \in \mathbb{N}} P_{\nu_m}^m, \quad \nu \in \mathcal{F} \tag{3.21}$$

form an orthonormal basis of $L^2(\Gamma, \Sigma, \mu)$, which is called generalized polynomial chaos basis. *If $\mu_m = \mathcal{N}_1$ for all $m \in \mathbb{N}$, then $(P_n^m)_{n \in \mathbb{N}_0}$ are hermite polynomials, interpreted as a basis of $L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \gamma)$.*

Example 3.7. *Consider μ_m as uniform distribution on $\Gamma_m := [-1, 1]$ for $m \in \mathbb{N}$, $\mu(d\xi) = \frac{1}{2} d\xi$. The corresponding orthonormal polynomials defined by three-term recursion*

$$\frac{n+1}{\sqrt{2n+3}\sqrt{2n+1}} L_{n+1}(\xi) = \xi L_n(\xi) - \frac{n}{\sqrt{2n+1}\sqrt{2n-1}} L_{n-1}(\xi), \tag{3.22}$$

where $L_{-1}(\xi) = 0$, $L_0(\xi) = 1$. These polynomials satisfy the Rodriguez formula

$$L_n(\xi) = \frac{\sqrt{2n+1}}{2^n n!} \frac{d^n}{d\xi} (\xi^2 - 1), \quad n \in \mathbb{N}_0. \tag{3.23}$$

The first few are $L_0(\xi) = 1$, $L_1(\xi) = \sqrt{3}\xi$, $L_2(\xi) = \frac{\sqrt{5}}{2}(3\xi^2 - 1)$. The measure space (Γ, Σ, μ) is a countable product of identical factors

$$([-1, 1], \mathcal{B}([-1, 1]), \frac{1}{2} d\xi), \tag{3.24}$$

$\Gamma = [-1,1]^\infty$, $\Sigma = \mathcal{B}([-1,1])^\infty = \mathcal{B}([-1,1]^\infty)$, $\mu = \bigotimes_{m\in\mathbb{N}} \mu_m$ with $\mu_m(d\xi) = \frac{1}{2}d\xi$, for $m \in \mathbb{N}_0$. The Legendre chaos basis $(L_\nu)_{\nu\in\mathcal{F}}$ is orthonormal basis of $L^2([-1,1]^\infty, \mathcal{B}([-1,1]^\infty), \mu)$.

Note that, even thought each μ_m is absolutely continuous w.r.t to the Lebesgue measure, the product density is zero. Moreover, there is no Lebesgue measure in infinite dimensions. Nevertheless, by Kolmogorovs extension theorem, we obtain well-posedness of the formulation above.

4 PDEs with uniform stochastic parameters

4.1 Parametric stochastic operators

Let V be a separable Hilbert space with dual space V^* and the (V^*, V) duality paring $\langle\cdot,\cdot\rangle$. We consider a parametric operator equation of the form

$$Au = f \tag{4.1}$$ 9

with $f \in V^*$ and $A \in \mathcal{L}(V, V^*)$ a linear, bounded operator from V to V^*.

Theorem 4.1. *If A is boundedly invertible, then* (4.1) *has a unique solution $u = A^{-1}f$.*

Let Γ be a topological space, A a parametric operator from V to V^* defined by the continuous map

$$A \colon \Gamma \to \mathcal{L}(V, V^*). \tag{4.2}$$

We assume that $A(y)$ is boundedly invertible, for all $y \in \Gamma$ and consider the parametric operator equation

$$A(y)u(y) = f(y), \qquad \forall y \in \Gamma \tag{4.3}$$ 10

for a map $f \colon \Gamma \to V^*$.

Proposition 4.2. *Eq* (4.3) *has a unique solution $u \colon \Gamma \to V$. It is continuous if and only if $f \colon \Gamma \to V^*$ is continuous.*

Proof. Existence follows directly and continuity of the map $y \mapsto A^{-1}(y)$ yields the claim. □

To derive the weak formulation of (4.3) in the parameter y, we additionally assume $A(y)$ is symmetric, positive definite for all $y \in \Gamma$ and there exists constants $\check{c}$ and $\hat{c}$, s.t.

$$\|A(y)\|_{V\to V^*} \le \check{c}, \quad \|A^{-1}(y)\|_{V^*\to V} \le \hat{c}, \quad \forall y \in \Gamma, \tag{4.4}$$ 11

i.e. the bilinear form $\langle A(y)\cdot,\cdot\rangle$ is a scalar product on V that induces a norm equivalent to $\|\cdot\|_V$. In the future we will omit the operator norm index. The estimates (4.4) always hold for Γ compact. Let μ be a probability measure on Borel-measurable set $(\Gamma, \mathcal{B}(\Gamma))$. Then, the operator $A(y)$ becomes stochastic since it depends on the parameter $y \in \Gamma$ in the probability space $(\Gamma, \mathcal{B}(\Gamma), \mu)$. Similarly, if f is a random variable random on $(\Gamma, \mathcal{B}(\Gamma), \mu)$, with values in V^*. We assume in the following

$$f \in L^2(\Gamma, \mathcal{B}(\Gamma); V^*). \tag{4.5}$$ 12

The linear variational problem will be obtained by multiplication of (4.3) by a test function $v\colon \Gamma \to V$ and integration over Γ,

$$\int_{\Gamma} \langle A(y)u(y), v(y)\rangle \mu(dy) = \int_{\Gamma} \langle f(y), v(y)\rangle \mu(dy). \tag{4.6}$$ 13

Theorem 4.3. *With* (4.4) *and* (4.5)*, the solution* u *of* (4.3) *is the unique element of* $L^2(\Gamma, \mathcal{B}(\Gamma), \mu; V)$ *satisfying* (4.6) *for all* $v \in L^2(\Gamma, \mathcal{B}(\Gamma), \mu; V)$*. Moreover,*

$$\|u\|_{L^2(\Gamma,\mathcal{B}(\Gamma),\mu;V)} \leq \check{c}\|f\|_{L^2(\Gamma,\mathcal{B}(\Gamma),\mu;V^*)}. \tag{4.7}$$

Remark 4.4. For separable Hilbert space X, the Lebesgue-Bochner space $L^2(\Gamma, \mathcal{B}(\Gamma), \mu; X)$ is isometrically isomorphic to the Hilbert tensor product space $X \otimes L^2(\Gamma, \mathcal{B}(\Gamma), \mu)$. In particular, the solution u can be seen as an element of $V \otimes L^2(\Gamma, \mathcal{B}(\Gamma), \mu)$ and f is an element of $V^* \otimes L^2(\Gamma, \mathcal{B}(\Gamma), \mu)$.

4.2 Stationary diffusion with stochastic coefficient

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$ and a probability space $(\Omega, \Sigma, \mathbb{P})$. The model problem reads as follows:

$$-\nabla \cdot (a(x,\omega)\nabla U(x,\omega)) = f(x), \qquad x \in D,\ \omega \in \Omega, \tag{4.8}$$ MP

$$U(x,\omega) = 0, \qquad x \in \partial D,\ \omega \in \Omega. \tag{4.9}$$

We assume uniform boundedness of the coefficient, i.e. for all $x \in D$ and $\omega \in \Omega$ there exists a $\check{a}, \hat{a} \in \mathbb{R}_{>0}$, s.t.

$$0 < \check{a} \leq a(x,\omega) \leq \hat{a} < \infty, \tag{4.10}$$ 15

and select some deterministic approximation $a_0 \in L^\infty(D)$ to the stochastic coefficient $a(x,\omega)$, e.q.

$$a_0(x) = \bar{a}(x) := \int_{\Omega} a(x,\omega) d\mathbb{P}(\omega) \tag{4.11}$$

or

$$a_0 = \frac{1}{2}(\check{a} + \hat{a}), \text{ or } a_0 = \sqrt{\check{a}\hat{a}}. \tag{4.12}$$

We consider a series expansion of $a(x,\omega) - a_0(x)$ which in case of $a_0 = \bar{a}$ is the fluctuation around the mean value. Let $(\varphi_m)_{m\in\mathbb{N}}$ be a biorthogonal basis of $L^2(D)$ with associated dual basis $(\tilde{\varphi}_m)_{m\in\mathbb{N}} \subset L^2(D)$, i.e.

$$\langle \varphi_m, \tilde{\varphi}_n\rangle = \delta_{m,n}, \tag{4.13}$$

and for $v \in L^2(D)$,

$$v = \sum_{m\geq 1} \langle v, \tilde{\varphi}_m\rangle \varphi_m \tag{4.14}$$ 16

with unconditional convergence. For a positive sequence $(\alpha_m)_{m\in\mathbb{N}}$, define RVs

$$Y_m(\omega) := \frac{1}{\alpha_m}\int_D (a(x,\omega) - a_0(x))\tilde{\varphi}_m(x)dx, \quad m \in \mathbb{N}. \tag{4.15}$$

By (4.14), for all $\omega \in \Omega$,

$$a(x,\omega) = a_0(x) + \sum_{m=1}^{\infty} \alpha_m \varphi_m(x) Y_m(\omega) \tag{4.16}$$

with unconditional convergence in $L^2(D)$.

lemma 1 **Lemma 4.5.** *There is a positive sequence* $(\alpha_m)_{m\in\mathbb{N}}$*, s.t.* $Y_m(\omega) \in [-1,1]$*, for all* $\omega \in \Omega$*.*

Proof. By Hölders inequality,

$$|\int_D (a(x,\omega) - a_0(x))\tilde{\varphi}_m dx| \le \|a(\cdot,\omega) - a_0(\cdot)\|_{L^\infty(D)} \|\tilde{\varphi}_m\|_{L^1(D)}. \tag{4.17}$$

Hence,

$$\alpha_m := \sup_{\omega\in\Omega} \|a(\cdot,\omega) - a_0(\cdot)\|_{L^\infty(D)} \|\tilde{\varphi}_m\|_{L^1(D)}. \tag{4.18}$$

□

Define as a parametric domain the compact and topological space

$$\Gamma = [-1,1]^\infty = \bigtimes_{m=1}^{\infty} [-1,1] \tag{4.19}$$

and let $(\alpha_m)_{m\in\mathbb{N}}$ be a sequence as in lemma 4.5. Assume that $\sum_{m=1}^\infty \alpha_m|\varphi_m(X)|$ converges in $L^\infty(D)$, i.e.

$$\lim_{M\to\infty} \operatorname*{ess\,sup}_{x\in D} \sum_{m=M}^{\infty} \alpha_m|\varphi_m(x)| = 0 \tag{4.20}$$

Then,

$$a_\varphi(x,y) := a_0(x) + \sum_{m=1}^{\infty} \alpha_m\varphi_m(x)y_m, \quad (y_m)_{m\in\mathbb{N}} \in \Gamma \tag{4.21}$$

converges unconditionally in $L^\infty(D)$ and the stochastic diffusion coefficient satfisfies

$$a(x,\omega) = a_\varphi(x, Y(\omega)), \quad x \in D,\, \omega \in \Omega, \tag{4.22}$$ 17

where $Y(\omega) := (Y_m(\omega))_{m\in\mathbb{N}} \in \Gamma$. Define the operators (with dual paring index ${}_{H^{-1}}\langle\cdot,\cdot\rangle_{H_0^1}$ omitted)

$$\langle A(y)v, w\rangle := \int_D a_\varphi(x,y)\nabla v(x)\cdot\nabla w(x)dx, \quad y \in \Gamma \tag{4.23}$$

$$\langle A_0 v, w\rangle := \int_D a_0(x)\nabla v(x)\cdot\nabla w(x)dx, \tag{4.24}$$

$$\langle A_m v, w\rangle := \int_D \alpha_m\varphi_m(x)\nabla v(x)\cdot\nabla w(x)dx, \quad m \in \mathbb{N} \tag{4.25}$$

for $v, w \in H_0^1(D)$. $A(y)$ is the operator associated to (4.8) for all $\omega \in \Omega$, namely

$$A(y) = A_0 + \sum_{m=1}^{\infty} A_m y_m, \quad y \in \Gamma. \tag{4.26}$$ 18

Hence, $U(x,\omega) = u(x, Y(\omega))$, for $\omega \in \Omega$ and $x \in D$. Note that (4.26) converges in $\mathcal{L}(H_0^1(D), H^{-1}(D))$ uniformly in $y \in \Gamma$ and $A(y)$ depends continuously on $y \in \Gamma$. We assume that the bilinear form associated to the operator A_0 is *coercive* on $H_0^1(D)$, i.e. there exists an $\check{a}_0$,s.t.

$$\operatorname*{ess\,inf}_{x\in D} a_0(x) \ge \check{a}_0 > 0. \tag{4.27}$$ 19

prop2 **Proposition 4.6.** *If*

$$\gamma := \frac{1}{\check{a}_0} \operatorname*{ess\,sup}_{x\in D} \sum_{m\geq 1} |\varphi_m(x)| < 1, \tag{4.28}$$ 20

then,

$$A(y)\colon H_0^1(D) \to H^{-1}(D) \tag{4.29}$$

is boundedly invertible for all $y \in \Gamma$ with

$$\sup_{y\in\Gamma} \|A^{-1}(y)\| \leq \frac{1}{\check{a}_0(1-\gamma)} \tag{4.30}$$

and $A(y)$ is bounded with

$$\sup_{y\in\Gamma} \|A(y)\| \leq \|a_0\|_{L^\infty(D)}(1+\gamma). \tag{4.31}$$

Proof. $A_0\colon H_0^1(D) \to H^{-1}(D)$ is invertible due to (4.27) and the Lax-Milgram lamma. The norm if A_0^{-1} is denoted by $\check{a}_0^{-1}$. By (4.28) we have

$$\|A_0^{-1}(A_0 - A(y))\| \leq \check{a}_0^{-1} \operatorname*{ess\,sup}_{x\in D} \sum_{m=1}^{\infty} \alpha_m |\varphi_m(x)| = \gamma < 1. \tag{4.32}$$

Therefore,

$$I - A_0^{-1}(A_0 - A(y)) = A_0^{-1}A(y) \tag{4.33}$$

is invertible by a Neumann series and has norm less than $(1-\gamma)^{-1}$. Multiplication by A_0 yields the statement. □

4.3 Discretization in Legendre polynomials

For a seperable Hilbert space V, consider the parametric operator in $\mathcal{L}(V, V^*)$ of the form

$$A(y) = A_0 + \sum_{m\geq 1} A_m y_m, \quad y \in [-1,1]^\infty \tag{4.34}$$ 21

with $A_0, A_m \in \mathcal{L}(V, V^*)$ and uniformly convergent in $\mathcal{L}(V, V^*)$ with $A(y)$ positive boundedly invertible for all y and continuous dependence on $y \in \Gamma = [-1,1]^\infty$, which holds by proposition 4.6. Let μ on $(\Gamma, \mathcal{B}(\Gamma))$ be a countable product of uniform measures on $[-1,1]$. The tensor product Legendre polynomials $(L_\nu)_{\nu\in\mathcal{F}}$ form an orthonormal basis of $L^2(\Gamma, \mathcal{B}(\Gamma), \mu)$. The operator $A(y)$ induces a boundedly invertible operator between the Hilbert tensor product spaces $V \otimes L^2(\Gamma, \mathcal{B}(\Gamma), \mu))$ and $V^* \otimes L^2(\Gamma, \mathcal{B}(\Gamma), \mu)$.

Definition 4.7. We introduce the *multiplication operator*, for $m \in \mathbb{N}$,

$$M_{y_m}\colon L^2(\Gamma, \mathcal{B}(\Gamma), \mu) \to L^2(\Gamma, \mathcal{B}(\Gamma), \mu),\ g(y) \mapsto y_m g(y). \tag{4.35}$$ 22

Remark 4.8. From $y_m \in [-1,1]$ it follows that M_{y_m} is self-adjoint and

$$\|M_{y_m}\| = 1, \quad \text{for all } m \in \mathbb{N}. \tag{4.36}$$ 23

prop3 **Proposition 4.9.** *We assume that*

$$\sum_{m\geq 1} \|A_m\| < \infty. \tag{4.37}$$

Then, the operator induced by $A(y)$ *in* (4.6) *can be represented by*

$$\mathcal{A} = A_0 \otimes I + \sum_{m=1}^{\infty} A_m \otimes M_{y_m} \tag{4.38}$$ 24

and $\mathcal{A} \in \mathcal{L}(V \otimes L^2(\varGamma, \mathcal{B}(\varGamma), \mu; V^* \otimes L^2(\varGamma, \mathcal{B}(\varGamma), \mu))$. (4.38) *converges unconditionally.*

Proof. The operator $\mathcal{A}$ is well-defined by (4.38), since by (4.36)

$$\|\sum_{m=M}^{\infty} A_m \otimes M_{y_m}\| \leq \sum_{m=M}^{\infty} \|A_m\|\|M_{y_m}\| = \sum_{m=M}^{\infty} \|A_m\|, \tag{4.39}$$

which can be made arbitrarily small for M large. Let $g \in L^2(\varGamma, \mathcal{B}(\varGamma), \mu)$ and $v \in V$. Then, with (4.38),

$$\mathcal{A}(v \otimes g)(y) = A_0 v g(y) + \sum_{m=1}^{\infty} A_m v y_m g(y) = A(y)(vg(y)), \quad y \in \varGamma. \tag{4.40}$$

Hence, $\mathcal{A}$ is the operator induced by A. □

Since, $(L_\nu)_{\nu \in \mathcal{F}}$, are a CONS of $L^2(\varGamma, \mathcal{B}(\varGamma), \mu)$,

$$T_L \colon l^2(\mathcal{F}) \to L^2(\varGamma, \mathcal{B}(\varGamma), \mu),\ (c_\nu)_{\nu \in \mathcal{F}} \mapsto \sum_{\nu \in \mathcal{F}} c_\nu L_\nu \tag{4.41}$$ 25

is an unitary isomorphism by Parseval's identity. The tensorization with the dual identity $I_V \colon V \to V^*$ on V yields the isometric isomorphism

$$I_V \otimes T_L \colon V \otimes l^2(\mathcal{F}) \to V^* \otimes L^2(\varGamma, \mathcal{B}(\varGamma), \mu), \tag{4.42}$$ 26

with adjoint

$$(I_V \otimes T_L)^* = I_{V^*} \otimes T_L^* \colon V^* \otimes L^2(\varGamma, \mathcal{B}(\varGamma), \mu) \to V \otimes l^2(\varGamma, \mathcal{B}(\varGamma), \mu) \tag{4.43}$$

Definition 4.10. The *semidiscrete operator* is given by

$$\mathcal{A}_L := (I_V \otimes T_L)^* \mathcal{A} (I_V \otimes T_L) \tag{4.44}$$ 27

and interpreting $f \in L^2(\varGamma, \mathcal{B}(\varGamma, \mu; V^*)$ as an element of $V^* \otimes L^2(\varGamma, \mathcal{B}(\varGamma), \mu)$,

$$F_L := (I_V \otimes T_L)^* f = \int_\varGamma f(y) L_\nu(y) \mu(dy), \quad \nu \in \mathcal{F}. \tag{4.45}$$

With the semidiscrete operator we obtain the *semidiscrete operator equation*

$$\mathcal{A}_L u_L = f_L \tag{4.46}$$ 28

with

$$\mathcal{A}_L = A \otimes I + \sum_{m\geq 1} A_m \otimes K_m, \tag{4.47}$$

$$K_m := T_L^* M_{y_m} T_L \tag{4.48}$$ 29

and

$$u = (I_V \otimes T_L) u_L. \tag{4.49}$$

lemma2 **Lemma 4.11.** *$K_m\colon l^2(\mathcal{F}) \to l^2(\mathcal{F})$, $m \in \mathbb{N}$ has the form*

$$K_m(c_\nu)_{\nu\in\mathcal{F}} = (\beta_{\nu_m+1}c_{\nu+\epsilon_m} + \beta_{\nu_m}c_{\nu-\epsilon_m})_{\nu\in\mathcal{F}} \tag{4.50}$$ 30

with $(\epsilon_m)_n = \delta_{m,n}$ and $\beta_n = \frac{1}{\sqrt{4-n^{-2}}} \in \left(\frac{1}{2}, \frac{1}{\sqrt{3}}\right]$, $n \in \mathbb{N}$. K_m is self-adjoint and $\|K_m\| = 1$, for every $m \in \mathbb{N}$.

Proof. Since, $T_L^{-1} = T_L^*$,

$$T_L K_m(c_\nu)_{\nu\in\mathcal{F}} = M_{y_m} T_L(L_\nu)_{\nu\in\mathcal{F}} = \sum_{\nu\in\mathcal{F}} c_\nu y_m L_\nu(y). \tag{4.51}$$

Therefore, (4.50) is equivalent to

$$y_m L_\nu(y) = \beta_{\nu_m+1}L_{\nu+\epsilon_m}(y) + \beta_{\nu_m}L_{\nu-\epsilon_m}(y). \tag{4.52}$$

The claim follows from the three term recursion

$$\xi L_n(\xi) = \beta_{n+1}L_{n+1}(\xi) + \beta_n L_{n-1}(\xi), \quad \xi \in [-1,1],\, n \in \mathbb{N}. \tag{4.53}$$

□

The solution u of (4.3) is

$$u(y) = \sum_{\nu\in\mathcal{F}} u_\nu L_\nu(y) \in V, \quad y \in \Gamma, \tag{4.54}$$

with convergence in $L^2(\Gamma, \mathcal{B}(\Gamma), \nu; V)$, where the coefficients $(u_\nu)_{\nu\in\mathcal{V}} \in V$ are determined by the equation

$$A_0 u_\nu + \sum_{m\geq 1} A_m(\beta_{\nu_m+1}u_{\nu_m+\epsilon_m} + \beta_{\nu_m}u_{\nu_m-\epsilon_m}) = f_\nu, \quad \nu \in \mathcal{F} \tag{4.55}$$

with $f_\nu := \int_\Gamma f(y)L_\nu(y)\mu(dy) \in V^*$.

4.4 full discretization with finite elements

Assume $\Lambda \subset \mathcal{F}$ with $|\Lambda| < \infty$ and let $V_{N,\nu} \subset V$ be finite dimensional spaces for $\nu \in \mathcal{F}$, with $V_{N,\nu} = \{0\}$ for $\nu \in \mathcal{F}$
Λ. Define

$$V_N := \{v \in L^2(\Gamma, \mathcal{B}(\Gamma), \mu; V)\colon V_\nu \in V_{N,\nu},\, \forall \nu \in \mathcal{F}\}. \tag{4.56}$$

where $v_\nu \in V$ is the ν^{th} coefficient of the expansion of $v \in L^2(\Gamma, \mathcal{B}(\Gamma), \mu; V)$ w.r.t $(L_\nu)_{\nu\in\mathcal{F}}$). This span can be interpreted as a subspace of $L^2(\Gamma; V)$ or as the span of sequences $(v_\nu)_{\nu\in\mathcal{F}}$ in V with $v_\nu \in V_{N,\mu}$ for $\nu \in \mathcal{F}$, which is a subspace of $l^2(\mathcal{F}; V)$. By Parseval's identity, the norms induced by these two spaces coincide.

5 PDEs with Gaussian parameters

We consider the Model Problem 4.8 with expansion of $\log(a-a_*)$ where $a_*(x) \geq 0$ bounded on the Lipschitz domain $D \subset \mathbb{R}^d$. The coefficient has the form

$$a(x,y) = a_*(x) + a_0 \exp\left(\sum_{m\geq 1} a_m(x) y_m\right), \quad x \in D, \tag{5.1}$$ 31

for $y = (y_m)_{m\in\mathbb{N}} \in \mathbb{R}^\infty$ and the RV $Y = (Y_m)_{m\in\mathbb{N}}$, as in 4.22, being a sequence of independent standard Gaussian RVs. This is e.q. the case if $(a_m)_{m\in\mathbb{N}}$ are orthonormal in the Cameron-Martin space of the distribution of $\log(a-a_*)$. y is in the probability space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \gamma)$, with the product measure $\gamma = \bigotimes_{m=1}^\infty \mathcal{N}_1$. Unfortunately, the series in 5.1 may not converge for all $y \in \mathbb{R}^\infty$. Hence, assume $a_m \in L^\infty(D)$, for $m \in \mathbb{N}_0$ and there exists an $\check{a}_0$, such that

$$a_0(x) \geq \check{a}_0 > 0, \quad \text{for } x \in D \tag{5.2}$$

and

$$\sum_{m\geq 1} \|a_m\|_{L^\infty(D)} < \infty \tag{5.3}$$ 32

i.e. $\alpha_m := \|a_m\|_{L^\infty(D)}$, $m \in \mathbb{N}$, we have $(\alpha_m)_{m\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Define

$$\Gamma := \{y \in \mathbb{R}^\infty \colon \sum_{m\geq 1} \alpha_m y_m\} \tag{5.4}$$ 33

Then 5.1 converges for $y \in \Gamma$.

lemma 3 **Lemma 5.1.** *$\Gamma \in \mathcal{B}(\mathbb{R}^\infty)$ and $\gamma(\Gamma) = 1$.*

Proof. The Borel-measureability follows from

$$\Gamma = \cup_{N=1}^\infty \cap_{M=1}^\infty \{y \in \mathbb{R}^\infty \colon \sum_{m=1}^M \alpha_m |y_m| \leq N\} \tag{5.5}$$

with

$$\int_{\mathbb{R}^\infty} |y_m| \gamma(dy) = \frac{2}{2\pi} \int_0^\infty \xi \exp\left(-\frac{\xi^2}{2}\right) d\xi = \sqrt{\frac{2}{\pi}} \tag{5.6}$$

it follows by monotone convergence that

$$\int_{\mathbb{R}^\infty} \sum_{m\geq 1} \alpha_m |y_m| \gamma(dy) = \sum_{m=1}^\infty \alpha_m \int_{\mathbb{R}^\infty} |y_m| \gamma(dy) = \sqrt{\frac{2}{\pi}} \sum_{m\geq 1} \alpha_m < \infty. \tag{5.7}$$

Hence, the sum converges γ-almost everywhere on $\mathbb{R}^\infty$. Hence, Γ exhibits full measure under γ. □

lemma 4 **Lemma 5.2.** *5.1 is well defined for all $y \in \Gamma$ and*

$$0 < \check{a}(y) := \operatorname*{ess\,inf}_{x\in D} a(x,y) \leq a(x,y) \leq \operatorname*{ess\,sup}_{x\in D} a(x,y) =: \hat{a}(y) < \infty. \tag{5.8}$$ 34

with

$$\hat{a}(y) \leq \|a_*\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \exp\left(\sum_{m\geq 1} \alpha_m |y_m|\right), \tag{5.9}$$

and

$$\check{a}(y) \geq \operatorname*{ess\,inf}_{x\in D} a_*(x) + \check{a}_0(y) \exp\left(-\sum_{m\geq 1} \alpha_m |y_m|\right). \tag{5.10}$$

Proof. Let $y \in \mathit{\Gamma}$ and $x \in D$ with $|a_m(x)| \leq \alpha_m$ for $m \in \mathbb{N}$. Then the claim follows by

$$\exp\left(\sum_{m\geq 1} a_m(x) y_m\right) = \prod_{m\geq 1} \exp\left(a_m(x) y_m\right) \in (0, \infty) \tag{5.11}$$

and the definition of $a(x, y)$ in 5.1. □

Due to 5.1 and 5.2 we consider $\mathit{\Gamma}$ as parameter space instead of $\mathbb{R}^\infty$. Note that $\mathit{\Gamma}$ is not a product domain but a product measure such as γ can be defined on $\mathit{\Gamma}$ by restriction.

5.1 Pathwise Properties

For $y \in \mathit{\Gamma}$, consider the weak form of the model problem 4.8 on $V := H_0^1(D)$ with norm

$$\|v\|_V^2 := \int_D |\nabla v(x)|^2 dx, \tag{5.12}$$

and solution $u(y) \in V$, s.t., for all $v \in V$,

$$\int_D a(x, y) \nabla u(x) \cdot \nabla v(x) dx = \int_D f(x, y) v(x) dx =: f(x; y) \tag{5.13}$$ logVar

It holds (by continuity and coercivity follows Lax-Milgram)

$$\|u(y)\| \leq \frac{1}{\check{a}(y)} \|f(\cdot, y\|_{V^*}, \quad y \in \mathit{\Gamma}. \tag{5.14}$$ reg

However, this is not satisfied uniformly since $a(x, y)$ is not bounded uniformly.

5.2 Review of Gaussian measures

For sequence $\sigma = (\sigma_m)_{m\in\mathbb{N}} \in \exp\left(\ell^1(\mathbb{N})\right)$, i.e. $\sigma_m = \exp(s_m)$, with $(s_m)_{m\in\mathbb{N}} \in \ell^1(\mathbb{N})$, define the product measure

$$\gamma_\sigma := \bigotimes_{m=1}^\infty \mathcal{N}_{\sigma_m^2} \tag{5.15}$$ 35

on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$.

prop 4 **Proposition 5.3.** *For $\sigma = (\sigma_m)_{m\in\mathbb{N}} \in \exp\left(\ell^1(\mathbb{N})\right)$, the measure γ_σ is equivalent to γ. The density of γ_σ w.r.t γ is*

$$\zeta_\sigma(y) := \left(\prod_{m=1}^\infty \frac{1}{\sigma_m}\right) \exp\left(-\frac{1}{2} \sum_{m=1}^\infty (\sigma_m^{-2} - 1) y_m^2\right). \tag{5.16}$$ 36

Proof. Let

$$\zeta_{\sigma,m} := \frac{1}{\sigma_m} \exp\left(-\frac{1}{2}(\sigma_m^{-1} - 1)y_m^2\right). \tag{5.17}$$

Then

$$d\mathcal{N}_{\sigma_m^2} = \zeta_{\sigma,m} d\mathcal{N}_1, \tag{5.18}$$

and

$$\int_{\mathbb{R}} (\zeta_{\sigma,m}(y_m))^{\frac{1}{2}} \mathcal{N}_1(dy_m) = \frac{1}{\sqrt{2\pi\sigma_m}} \int_{\mathbb{R}} \exp\left(-\frac{1}{4}(\sigma_m^{-2} - 1)y_m^2\right) dy_m \tag{5.19}$$

$$= \left(\frac{2}{\sigma_m + \sigma_m^{-1}}\right)^{\frac{1}{2}} = \exp\left(\frac{1}{2}\beta_m\right) \tag{5.20}$$

for some β_m with $|\beta_m| \le \log(\sigma_m)$. Tensorization yields

$$\prod_{m=1}^{\infty} \int_{\mathbb{R}} (\zeta_{\sigma,m}(y_m))^{\frac{1}{2}} \mathcal{N}_1(dy) = \exp\left(\frac{1}{2}\sum_{m\ge 1} \beta_m\right), \tag{5.21}$$

which converges, since $(\log(\sigma_m))_{m\in\mathbb{N}}$ converges is $\ell^1(\mathbb{N})$. The claims follows by the theorem of Kakutani. See Da Prato 2006. □

Proposition 5.3 implies $\gamma_\sigma(\Gamma) = 1$ for any $\sigma \in \exp\left(\ell^1(\mathbb{N})\right)$ and the restriction of γ_σ onto Γ is a probability measure. Consider the sequences $\sigma = (\sigma_m)_{m\in\mathbb{N}}$ with

$$\sigma_m(\chi) := \exp(\chi\alpha_m), \quad m \in \mathbb{N}, \chi \in \mathbb{R} \tag{5.22}$$

and let $\gamma_\chi := \gamma_{\sigma(\chi)}$, $\zeta_\chi := \zeta_{\sigma(\chi)}$. Note that $\gamma_\chi = \gamma$ for $\chi = 0$.

lemma 5 **Lemma 5.4.** *Let $\eta < \chi$ and $k \ge 0$. Then, for $y \in \Gamma$,*

$$\frac{\zeta_\eta(y)}{\zeta_\chi(y)} \exp\left(k \sum_{m\ge 1} \alpha_m |y_m|\right) \le \exp\left(\left(\frac{k^2 e^{2\chi\|\alpha\|_{\ell^\infty(\mathbb{N}}}}{4(\chi-\eta)} + \chi - \eta\right) \|\alpha\|_{\ell^1(\mathbb{N})}\right). \tag{5.23}$$

For $k = 0$,

$$\frac{\zeta_\eta(y)}{\zeta_\chi(y)} \le \exp\left((\chi - \eta)\|\alpha\|_{\ell^1(\mathbb{N})}\right). \tag{5.24}$$ 39

Proof. Let $y \in \Gamma$ and $\sigma_m := e^{\alpha_m}$. By 5.16 we have

$$\frac{\zeta_\eta(y)}{\zeta_\chi(y)} = \left(\prod_{m=1}^{\infty} \sigma_m^{\chi-\eta}\right) \exp\left(-\frac{1}{2}\sum_{m=1}^{\infty} (\sigma_m^{-2(\chi-\eta)} - 1)\sigma_m^{-2\eta} y_m^2\right). \tag{5.25}$$

With the estimate

$$\left(\sigma_m^{-2(\chi-\eta)} - 1\right)\sigma_m^{-2\eta} = \left(e^{-2(\chi-\eta)\alpha_m} - 1\right) e^{-2\eta\alpha_m} \tag{5.26}$$

$$= e^{-2\chi\alpha_m}\left(1 - e^{2(\chi-\eta)\alpha_m}\right) \tag{5.27}$$

$$\le e^{-2\chi\alpha_m}\left(-(\chi-\eta)\alpha_m\right) \tag{5.28}$$

we obtain

$$\log\left(\frac{\zeta_\eta(y)}{\zeta_\chi(y)}\exp\left(k\sum_{m\geq 1}\alpha_m|y_m|\right)\right) \tag{5.29}$$

$$= k\sum_{m\geq 1}\alpha_m|y_m| + \frac{1}{2}\sum_{m\geq 1}(\sigma_m^{-2(\chi-\eta)-1}\sigma_m^{-2\eta}y_m^2 + (\chi-\eta)\sum_{m\geq 1}\log(\sigma_m) \tag{5.30}$$

$$\leq k\sum_{m\geq 1}\alpha_m|y_m| - (\chi-\eta)\sum_{m\geq 1}\alpha_m e^{-2\chi\alpha_m}y_m^2 + (\chi-\eta)\sum_{m\geq 1}\alpha_m \tag{5.31}$$

$$= -\sum_{m\geq 1}\alpha_m\left(\sqrt{\chi-\eta}e^{-\chi\alpha_m}|y_m| - \frac{ke^{\chi\alpha_m}}{2\sqrt{\chi-\eta}}\right)^2 \tag{5.32}$$

$$+\sum_{m\geq 1}\frac{\alpha_m k^2 e^{2\chi\alpha_m}}{4(\chi-\eta)} + (\chi-\eta)\sum_{m\geq 1}\alpha_m \tag{5.33}$$

$$\leq \sum_{m\geq 1}\left(\frac{k^2e^{2\chi\alpha_m}}{4(\chi-\eta} + (\chi-\eta)\right)\alpha_m. \tag{5.34}$$

□

prop 5 **Proposition 5.5.** *Let $0 < p < \infty$ and $\eta < \chi$. Then*

$$L^p(\Gamma,\gamma_\chi) \subset L^p(\Gamma,\gamma_\eta) \tag{5.35}$$ 40

and

$$\|v\|_{L^p(\Gamma,\gamma_\eta)} \leq \exp\left(\frac{\chi-\eta}{p}\|\alpha\|_{\ell^1(\mathbb{N})}\|\right)\|v\|_{L^p(\Gamma,\gamma_\chi)}, \quad \forall v \in L^p(\Gamma,\gamma_\chi). \tag{5.36}$$

Proof.

$$\|v\|_{L^p(\Gamma,\gamma_\eta)} = \int_\Gamma v^p d\gamma_\eta = \int_G ammav^p\frac{\zeta_\eta}{\zeta_\chi}d\gamma_\chi \leq \sup_{y\in\Gamma}\frac{\zeta_\eta(y)}{\zeta_\chi(y)}\|v\|^p_{L^p(\Gamma,\gamma_\chi)}. \tag{5.37}$$

Then, the claim follows from Lemma 5.4 with $k = 0$. □

5.3 Integrability of solution

Proposition 5.6. *Let $0 < p < \infty$ and $\varrho > 0$. If $f \in L^p(\Gamma,\gamma_\varrho;V^*)$ then, the solution u of the variational problem 5.13 is in $L^p(\Gamma,\gamma;V)$ and*

$$\|u\|_{L^p(\Gamma,\gamma;V)} \leq \bar{c}_{\varrho,p}\|f\|_{L^p(\Gamma,\gamma_\varrho;V^*)} \tag{5.38}$$

with

$$\bar{c}_{\varrho,p} = \min\left\{\frac{\exp\left(\frac{\varrho}{p}\|\alpha\|_{\ell^1(\mathbb{N})}\right)}{\operatorname{ess\,inf}_{y\in\Gamma}a_*(y)}, \frac{1}{\check{a}_0}\exp\left(\left(\frac{pe^{2\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}}}{4\varrho} + \frac{\varrho}{p}\right)\|\alpha\|_{\ell^1(\mathbb{N})}\right)\right\}. \tag{5.39}$$

Proof. By 5.14 we have

$$\int_\Gamma \|u(y)\|_V^p\gamma(dy) \leq \int_\Gamma \zeta_\varrho^{-1}\check{a}(y)^{-p}\|f(\cdot,y)\|^p_{V^*}\gamma_\varrho(dy). \tag{5.40}$$

The claim follows from Lemma 5.2 and Lemma 5.4, with $\eta = 0$, $\chi = \varrho$ and $k = p$. □

lemma 6 **Lemma 5.7.** *For $\varrho \geq 0$ and $0 < r < \infty$,*

$$\exp\left(\sum_{m=1}^{\infty} \alpha_m |y_m|\right) \in L^r(\Gamma, \gamma_\varrho) \tag{5.41}$$

with

$$\| \exp\left(\sum_{m=1}^{\infty} \alpha_m |y_m|\right) \|_{L^r(\Gamma,\gamma_\varrho)} \leq \exp\left(\frac{r}{2} e^{2\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}} \|\alpha\|^2_{\ell^2(\mathbb{N})} + \sqrt{\frac{2}{\pi}} e^{\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}} \|\alpha\|_{\ell^1(\mathbb{N})}\right) \tag{5.42}$$

Proof. see Gittelson stochastic Galerkin 2010 Lemma 3.10. □

Thm 2 **Theorem 5.8.** *Let $0 < q < p < \infty$ and $\varrho \geq 0$. If $f \in L^p(\Gamma, \gamma_\varrho; V^*)$, then the solution u or 5.13 is in $L^q(\Gamma, \gamma_\varrho; V)$ and satisfies*

$$\|u\|_{L^q(\Gamma,\gamma_rho;V)} \leq \tilde{c}_{\varrho,p,q} \|f\|_{L^p(\Gamma,\gamma_\varrho;V^*)} \tag{5.43}$$

with

$$\tilde{c}_{\varrho,p,q} = \frac{1}{\check{a}_0} \exp\left\{\frac{qpe^{2\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}}}{2(p-q)} \|\alpha\|^2_{\ell^2(\mathbb{N})} + \sqrt{\frac{2}{\pi}} e^{\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}} \|\alpha\|_{\ell^1(\mathbb{N})}\right\} \tag{5.44}$$

Proof. Let $r = \frac{qp}{p-q}$. Then, by 5.14 and the Hö lder inequality we obtain

$$\int_\Gamma \|u(y)\|^q_V \gamma_\varrho(dy) \leq \int_\Gamma \check{a}(y)^{-q} \|f(\cdot;y)\|^q_{V^*} \gamma_\varrho(dy) \tag{5.45}$$

$$\leq \|\check{a}(\cdot)^{-1}\|^q_{L^r(\Gamma,\gamma_\varrho)} \|f\|^q_{L^p(\Gamma,\gamma_\varrho;V^*)}. \tag{5.46}$$

The claim follows from Lemma 5.2 and 5.4. □

If $f \in L^p(\Gamma, \gamma\varrho; V^*)$ with $p > 2$, then $u \in L^2(\Gamma, \gamma_\varrho; V)$ and

$$\|u\|_{L^2(\Gamma,\gamma_\varrho;V} \leq \tilde{c}_{\varrho,q} \|f\|_{L^p(\Gamma,\gamma_\varrho;V^*} \tag{5.47}$$

with

$$\tilde{c}_{\varrho,p} = \frac{1}{\check{a}_0} \exp\left\{\frac{pe^{2\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}}}{(p-2)} \|\alpha\|^2_{\ell^2(\mathbb{N})} + \sqrt{\frac{2}{\pi}} e^{\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}} \|\alpha\|_{\ell^1(\mathbb{N})}\right\} \tag{5.48}$$ const_rho_p

An orthonormal basis of $L^2(\Gamma, \gamma_\varrho)$ can be obtained by a transformation of the tensorized Hermite polynomials $(H_\nu)_{\nu \in \mathcal{F}}$.

lemma 7 **Lemma 5.9.** *For $\varrho \in \mathbb{R}$, the map*

$$L^2(\Gamma, \gamma) \to L^2(\Gamma, \gamma_\varrho), \quad v \mapsto v \circ \tau_\varrho, \tag{5.49}$$

with $\tau_\varrho \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$, $(y_m)_{m\in\mathbb{N}} \mapsto (e^{-\varrho\alpha_m} y_m)_{m\in\mathbb{N}}$, is a unitary isomoprhism of Hilbert spaces and

$$\int_\Gamma v(y)\gamma(dy) = \int_\Gamma v(\tau_\varrho(y))\gamma_\varrho(dy), \quad \forall v \in L^2(\Gamma, \gamma). \tag{5.50}$$ 40b

Proof. γ is the measure of γ_ϱ under the bijective map τ_ϱ, i.e.

$$\gamma(E) = \gamma_\varrho(\tau_\varrho^{-1}(E)) \tag{5.51}$$

for all $E \in \mathcal{B}(\Gamma)$. 5.50 is an application of the transformation theorem. □

Remark 5.10. $(H_\nu \circ \tau_\varrho)_{\nu\in\mathcal{F}}$ is an orthonormal basis of $L^2(\Gamma, \gamma_\varrho)$, by 5.9

Remark 5.11. Let $\varrho \geq 0$, $f \in L^p(\Gamma, \gamma_\varrho)$ with $p > 2$. Then, $u \in L^2(\Gamma, \gamma_\varrho; V)$ and

$$u(y) = \sum_{\nu\in\mathcal{F}} u_\nu H_\nu(\tau_\varrho(y)), \quad y \in \Gamma, \tag{5.52}$$ 41

with convergence in $L^2(\Gamma, \gamma_\varrho; V)$ and coefficients

$$u_\nu = \int_\Gamma u(\tau_\varrho^{-1}(y))H_\nu(y)\gamma(dy), \quad \nu \in \mathcal{F}. \tag{5.53}$$ 42

Furthermore, $\tilde{u} := (u_\nu)_{\nu\in\mathcal{F}} \in \ell^2(\mathcal{F}; V)$ and

$$\|\tilde{u}\|_{\ell^2(\mathcal{F};V)} \leq \tilde{c}_{\varrho,p}\|f\|_{L^p(\Gamma,\gamma_\varrho;V^*)} \tag{5.54}$$

with constant as in 5.48.

5.4 Weak formulation on problem-dependent spaces

Assume $0 \leq \vartheta < 1$ and $\varrho > 0$.

Definition 5.12. Define

$$B_{\vartheta\varrho}(w, v) := \int_\Gamma b(w(y), v(y); y)d\gamma_{\vartheta\varrho}(y) \tag{5.55}$$

$$= \int_\Gamma \int_D a(x, y)\nabla w(x, y) \cdot \nabla v(x, y)dxd\gamma_{\vartheta\varrho}(y) \tag{5.56}$$

and assuming $y \mapsto f(\cdot, y) \in V^*$ is $\mathcal{B}(\Gamma)$-measurable and sufficiently integrable, such that

$$F_{\vartheta\varrho}(v) := \int_\Gamma f(v(y); y)\gamma_{\vartheta\varrho}(dy) \tag{5.57}$$ 43

$$= \int_\Gamma \int_D f(x)v(x, y)dx\gamma_{\vartheta\varrho}(dy). \tag{5.58}$$

Furthermore, define the space

$$\mathcal{V}_{\vartheta\varrho} := \{v : \Gamma \to V, \ \mathcal{B}(\Gamma)\text{-measurable and } B_{\vartheta\varrho}(v, v) < \infty\}. \tag{5.59}$$ 44

Note that, more precisely $\mathcal{V}_{\vartheta\varrho}$ contains equivalence classes of $\gamma_{\vartheta\varrho}$-a.e. identical functions.

Remark 5.13. $\mathcal{V}_{\vartheta\varrho}$ becomes a Hilbert space when endowed with the inner product $B_{\vartheta\varrho}(\cdot, \cdot)$.

lemma 8 **Lemma 5.14.**

(i) *For* $v, w \in L^2(\Gamma, \gamma_\varrho; V)$,

$$|B_{\vartheta\varrho}(w, v)| \leq \hat{c}_{\vartheta\varrho} \|w\|_{L^2(\Gamma,\gamma_\varrho;V)} \|v\|_{L^2(\Gamma,\gamma_\varrho;V)} \tag{5.60}$$

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with

$$\hat{c}_{\vartheta\varrho} = \left(\|a_*\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \exp\left(\frac{e^{2\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}}}{4(1-\vartheta)\varrho} \|\alpha\|_{\ell^1(\mathbb{N})} \right) \right) \exp\left((1-\vartheta)\varrho\|\alpha\|_{\ell^1(\mathbb{N})} \right). \tag{5.61}$$

(ii) *For* $v \in L^2(\Gamma, \gamma; V)$ *with* $B_{\vartheta\varrho}(v, v) < \infty$,

$$V_{\vartheta\varrho}(v, v) \geq \check{c}_{\vartheta\varrho} \|v\|_{L^2(\Gamma,\gamma;V)} \tag{5.62}$$

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with

$$\check{c}_{\vartheta\varrho} = \left(\operatorname*{ess\,inf}_{x\in D} a_*(x) + \check{a}_0 \exp\left(\frac{e^{2\vartheta\varrho\|\alpha\|_{\ell^\infty(\mathbb{N})}}}{4\vartheta\varrho} \|\alpha\|_{\ell^1(\mathbb{N})} \right) \right) \exp\left(-\vartheta\varrho\|\alpha\|_{\ell^1(\mathbb{N})} \right). \tag{5.63}$$

Proof. (i) For $y \in \Gamma$, by continuity of $b(\cdot, \cdot; y)$ we have

$$\|B_{\vartheta\varrho}(w, v)\| \leq \int_\Gamma \frac{\zeta_{\vartheta\varrho}(y)}{\zeta_\varrho(y)} \hat{a}(y) \|w(y)\|_V \|v(y)\|_V \gamma_\varrho(dy) \tag{5.64}$$

$$\leq \|\frac{\zeta_{\vartheta\varrho}}{\zeta_\varrho} \hat{a}\|_{L^\infty(\Gamma,\gamma)} \|w\|_{L^2(\Gamma,\gamma_\varrho;V)} \|v\|_{L^2(\Gamma,\gamma_\varrho;V)} \tag{5.65}$$

The claim follows from Lemma 5.2 and 5.4 with $\eta = \vartheta\varrho$, $\chi = \varrho$ and $k = 1$.

(ii) For $y \in \Gamma$, by coercivity of $b(\cdot, \cdot; y)$

$$B_{\vartheta\varrho}(v, v) \geq \int_\Gamma \zeta_{\vartheta\varrho}(y) \check{a}(y) \|v(y)\|_V \gamma(dy) \tag{5.66}$$

$$\geq \operatorname*{ess\,inf}_{y\in\Gamma} \zeta_{\vartheta\varrho}(y) \check{a}(y) \|v\|_{L^2(\Gamma,\gamma;V)}. \tag{5.67}$$

Then, by Lemma 5.2 and 5.4 the claim follows with $\eta = 0$, $\chi = \vartheta\varrho$ and $k = 1$. □

prop 7 **Proposition 5.15.** *For a weight* $\vartheta \geq 0$, *the Hilbert space* $\mathcal{V}_{\vartheta\varrho}$ *is related to the Lebesgue-Bochner space by contiunous embeddings, i.e.*

$$L^2(\Gamma, \gamma_\varrho; V) \subset \mathcal{V}_{\vartheta\varrho} \subset L^2(\Gamma, \gamma; V). \tag{5.68}$$

For $\vartheta = 0$, *this holds only for*

$$\operatorname*{ess\,inf}_{x\in D} a_*(x) > 0 \tag{5.69}$$

Proof. By Lemma 5.14, we have for all $v \in L^2(\Gamma, \gamma_\varrho; V)$

$$\check{c}_{\vartheta\varrho}|v\|_{L^2(\Gamma,\gamma;V)} \leq B_{\vartheta\varrho}(v, v) \leq \hat{c}_{\vartheta\varrho}\|v\|_{L^2(\Gamma,\gamma_\varrho;V)}. \tag{5.70}$$

□

From 5.24 with $\eta = \vartheta\varrho$ and $\chi = \varrho$, it follows that if $f \in L^2(\Gamma, \gamma_\varrho; V^*)$, then $F_{\vartheta\varrho}$ is in the dual $\mathcal{V}^*_{\vartheta\varrho}$.

Theorem 5.16. *If $F_{\vartheta\varrho} \in \mathcal{V}^*_{\vartheta\varrho}$, then the solution u of 5.13 is the unique solution in $\mathcal{V}_{\vartheta\varrho}$ of the linear variational problem*

$$B_{\vartheta\varrho}(u, v) = F_{\vartheta\varrho}(v), \quad \forall v \in \mathcal{V}_{\vartheta\varrho}. \tag{5.71}$$

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Moreover,

$$b(u, w; \cdot) = f(w; \cdot) \tag{5.72}$$

holds $\gamma_{\vartheta\varrho}$-a.e.

Proof. By the Riesz isomorphism on the Hilbert space $\mathcal{V}_{\vartheta\varrho}$, 5.71 has a unique solution $u \in \mathcal{V}_{\vartheta\varrho}$. Setting $v(y) = w1_E(y)$ for $E \in \mathcal{B}(\Gamma)$ on which $\check{a}(y)$ is bounded. It follows that the solution of 5.71 satisfies

$$\int_E b(u, w; y) - f(w; y)\gamma_{\vartheta\varrho}(dy) = 0. \tag{5.73}$$

Since Γ is a countable union of such sets E, the integrand must vanishes $\gamma_{\vartheta\varrho}$-a.e. on Γ. The claim follows since $w \in V$ arbitrary. □

6 Galerkin Approximation

Our aim is to define a fully discretized version of 5.71. Therefore, consider the finite dimensional space

$$\mathcal{V}_N := \{v \in L^2(\Gamma, \gamma_\varrho; V) \colon v_\nu \in V_{N,\nu},\, \nu \in \mathcal{F}\} \subset L^2(\Gamma, \gamma_\varrho; V) \subset \mathcal{V}_{\vartheta\varrho} \tag{6.1}$$

with $V_{N,\nu} = \{0\}$ for all but finitely many $\nu \in \Lambda \subset \mathcal{F}$ and $V_{N,\nu}$ e.q. finite element spaces on some mesh $\mathcal{T}$ of domain D for $\nu \in \Lambda$.

Remark 6.1. 5.71 with $u, v \in \mathcal{V}_N$ is well-definied with a unique solution, since $\mathcal{V}_N$ is a closed subspace of $\mathcal{V}_{\vartheta\varrho}$ and a Hilbert space with inner product $B_{\vartheta\varrho}(\cdot, \cdot)$.

theorem 4

Theorem 6.2. *If $f \in L^p(\Gamma, \gamma_{\vartheta\varrho}; V^*)$ for $p > 2$, then the Galerkin projection u_N satisfies*

$$\|u - u_N\|_{L^2(\Gamma,\gamma;V)} \leq \sqrt{\frac{\hat{c}_{\vartheta\varrho}}{\check{c}_{\vartheta\varrho}}} \|u - v_N\|_{L^2(\Gamma,\gamma_\varrho;V)} \tag{6.2}$$

48

Proof. Theorem 5.8 implies that $u \in L^2(\Gamma, \gamma_\varrho; V)$. By definition u_N is the orthogonal projection of u onto $\mathcal{V}_N$ with respect to $B_{\vartheta\varrho}(\cdot, \cdot)$. This Galerkin projection minimized the error in the energy norm induced by the inner product $B_{\vartheta\varrho}(\cdot, \cdot)$. Then, by Lemma 5.14 we have

$$\check{c}_{\vartheta\varrho}\|u - u_N\|^2_{L^2(\Gamma,\gamma;V)} \leq B_{\vartheta\varrho}(u - u_N, u - u_N) \tag{6.3}$$

$$= \inf_{v_N \in \mathcal{V}_N} B\vartheta\varrho(u - v_N, u - v_N) \tag{6.4}$$

$$\leq \hat{c}_{\vartheta\varrho} \inf_{v_N \in \mathcal{V}_N} \|u - v_N\|^2_{L^2(\Gamma,\gamma_\varrho;V)}. \tag{6.5}$$

□

Remark 6.3. The errors in 6.2 are measured in different norms. Hence, we call 6.2 an *almost quasi-optimality* result.

Remark 6.4. $\frac{\hat{c}_{\vartheta\varrho}}{\check{c}_{\vartheta\varrho}}$ tends to ∞ for $\varrho \to 0$ or $\vartheta \to \infty$ or $\vartheta \to 1$. $\operatorname{ess\,inf}_{x \in D} a_*(x) = 0$ and $\vartheta \to 0$.

6.1 Optimal convergence rates of stochastic Galerkin approximation

The considered Galerkin approximation of the solution of the parametric deterministic formulation of a stochastic problem is well-defined and quasi-optimal in mean-square sense. Our goals are

(i) Identification of the best possible convergence rate for an N-term polynomial chaos approximation (based on regularity of the solution in terms of summability of the chaos coefficients.)

(ii) Construction of an optimal index sets $\Lambda \subset \mathcal{F}$ of active polynomials, such that $|\lambda| \leq N$.

Remark 6.5. (i) can be approached similar to standard regularity theory of PDEs (bootstrapping arguments) or by (more modern) analytic continuation, which often yields sharper bounds.

6.1.1 Affine Case with Legendre Chaos

We assume that $Y_m(\omega) \in [-1, 1]$ for $\omega \in \Omega$, $m \in \mathbb{N}$, and denote the scaled coefficient functions of $a(x, \omega)$ by $\psi_j(x) = \alpha_j \varphi_j(x)$, for $j = 1, 2, \ldots$. Furthermore, assume the following additional conditions

49 **Assumption 6.6.**

$$\psi_j \in L^\infty(D), \quad j \in \mathbb{N} \tag{6.6}$$ C1

$$y = (y_1, \ldots) \in [-1, 1]^{\mathbb{N}} \tag{6.7}$$ C2

$$a(x, y) = a_0(x) + \sum_{j \geq 1} \psi_j(x) y_j \quad \textit{for all } x \in D \textit{ and } y \in \Gamma \tag{6.8}$$ C3

assumption 1 **Assumption 6.7** (UEA - uniform ellipticity assumption in $\mathbb{R}$)**.**
Assume there exists $0 < r \leq R < \infty$ *s.t. for all* $x \in D$

$$0 < r \leq a(x, y) \leq R < \infty, \quad \textit{UEA(r,R)}. \tag{6.9}$$

Remark 6.8. The uniform boundedness in assumption 6.7 yields directly boundedness for the coefficient field mean by taking $y_j = 0$ for $j \in \mathbb{N}$

$$r \leq a_0(x) \leq R \quad \text{for } x \in D. \tag{6.10}$$

Moreover the equivalent formulations

$$\sum_{j \geq 1} |\psi_j(x)| \leq a_0(x) - r \tag{6.11}$$ 50

and

$$\sum_{j \geq 1} |\psi_j(x)| \leq R - a_0(x). \tag{6.12}$$ 51

Assumption 6.9 (UEA in $\mathbb{C}$ with complex parameters y_j)**.**
Assume there exists $0 < r \leq R < \infty$ *s.t. for* $x \in D$*,* $z \in U \subset \mathbb{C}^{\mathbb{N}}$*,*

$$0 < r < Re(a(x, z)) \leq |a(x, z)| \leq R < \infty, \quad \textit{UEAC(r,R)} \tag{6.13}$$

with complex variables $z = (z_j)_{j=1,...}$ *and* $|z_j| \leq 1$ *for* $j \in \mathbb{N}$. *Hence, the parameter vector* z *belongs to the polydisc*

$$U := \prod_{j\geq 1}\{z_j \in \mathbb{C}\colon |z_j| \leq 1\} \supset \Gamma \tag{6.14}$$ 52

Remark 6.10. UEA(r,R) implies UEAC(r,2R) for real a_0 and y_j. Hence, solution $u(z)$ is well-definied in V for $z \in U$ y complex Lax-Milgram.

lemma 9 **Lemma 6.11** (Stechkin Lemma)**.** *Let* $(\gamma_n)_{n\geq 1}$ *be a decreasing sequence of non-decreasing integers. Then, for any* $0 < p \leq q < \infty$ *and* $N \in \mathbb{N}$,

$$\left(\sum_{n\geq N}\gamma_n^q\right)^{\frac{1}{q}} \leq N^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{n\geq 1}\gamma_n^p\right)^{\frac{1}{p}} \tag{6.15}$$ 53

Proof. For $q < \infty$, combine the estimations

$$\sum_{n\geq N}\gamma_n^q \leq \gamma_N^{q-p}\sum_{n\geq N}\gamma_n^p \leq \gamma_N^{q-p}\sum_{n\geq 1}\gamma_n^p \tag{6.16}$$

and

$$N\gamma_N^p \leq \sum_{n\leq N}\gamma_n^P \leq \sum_{n\geq 1}\gamma_n^p \tag{6.17}$$

□

6.11 will be used repeatedly although there are recent (sometimes) sharper techniques.

Definition 6.12. Let $\nu \in \mathcal{F}$ with support in $\{1,\dots,J\}$ and a complex sequence $\alpha = (\alpha_j)_{j\geq 1}$. Then, we define the partial derivative with respect to a multi-index

$$\partial^\nu u = \frac{\partial^{|\nu|}u}{\partial^{\nu_1}y_1\cdots\partial^{\nu_J}y_J} \tag{6.18}$$

and by convenience we set

$$\nu! := \prod_{j\geq 1}\nu_j! \tag{6.19}$$

$$0! = 1 \tag{6.20}$$

$$\alpha^\nu := \prod_{j\geq 1}\alpha_j^{\nu_j} \tag{6.21}$$

theorem 5 **Theorem 6.13.** *If* $a(x,y)$ *fulfills UEAC(r,R) for some* $0 < r \leq R < \infty$ *and if* $\left(\|\psi_j\|_{L^\infty(D)}\right)_{j\geq 1} \in \ell^p(\mathbb{N})$ *for some* $0 < p < 1$, *then* $u(z)$ *is analytic as a mapping from* U *into* V *and* $u(z)$ *can be expanded by a Taylor series*

$$u(z) = \sum_{\nu\in\mathcal{F}} t_\nu z^\nu \quad \text{in } V, \tag{6.22}$$ 54

where $t_\nu := \frac{1}{\nu!}\partial^\nu u(0)$ *for* $\nu \in \mathcal{F}$, *and* $t_\mu \in V$ *and* $(\|t_\nu\|_V)_{\nu\in\mathcal{F}} \in \ell^p(\mathcal{F})$. *The convergence of 6.22 is unconditional: if* $(\Lambda_N)_{N\geq 1}$ *is an increasing sequence of* *finite sets which exhausts $\mathcal{F}$, i.e. there exists an $N_0 \in \mathbb{N}$, s.t. any finite subset $\tilde{\Lambda} \subset \mathcal{F}$ is contained in all sets Λ_N for all $N \geq N_0$. Then, the partial sums*

$$S_{\Lambda_N} u(z) = \sum_{\nu \in \Lambda_N} t_\nu z^\nu \tag{6.23}$$

satisfy

$$\lim_{N\to\infty} \sup_{z\in U} \|u(z) - S_{\lambda_N} u(z)\|_V = 0. \tag{6.24}$$

If Λ_N is a set of $\nu \in \mathcal{F}$ corresponding to the indicies of N Taylor coefficients with largest norms $\|t_\nu\|_V$, it holds

$$\sup_{z\in U} \|u(z) - S_{\Lambda_N} u(z)\|_V \leq \|(\|t_\nu\|_V)_{\nu\in\mathcal{F}}\|_{L^p(\mathcal{F})} N^{-s} \tag{6.25}$$ 56

with $s = \frac{1}{p} - 1$.

Analyticity can e.q. be proved by a power series and bootstrapping argument, or by strong differentiability of u w.r.t. complex coordinates z_j. If UEAC(r,R) holds, then $z \mapsto u(z)$ is V-valued bounded analytic function in certain domains larger than U. To define U we choose $0 < \delta \leq 2R < \infty$, and consider the sets

$$\mathcal{A}_\delta := \{z \in \mathbb{C}^{\mathbb{N}} \colon \delta \leq \mathrm{Re}(a(x,z)) \leq |a(x,z)| \leq 2R \text{ for } x \in D\} \tag{6.26}$$ 57

and $\mathcal{A}_\delta$ contains U for $0 < \delta < r$. By Lax-Milgram, the unique solution $u(z) \in V$ for $z \in \mathcal{A}_\delta$ satisfies the a priori estimate

$$\|u(z)\|_V \leq \frac{\|f\|_{V^*}}{\delta} \tag{6.27}$$ 58

Holomorphy of $u(z)$ is shown by the difference quotient argument which uses a pertubation lemma starting from a stability result.

lemma 10 **Lemma 6.14.** *Let u and $\tilde{u}$ be solutions of 4.8 with UEAC(r,R) coefficients α and $\tilde{\alpha}$, then*

$$\|u - \tilde{u}\|_V \leq \frac{\|f\|_{V^*}}{r^2} \|\alpha - \tilde{\alpha}\|_{L^\infty(D)} \tag{6.28}$$ 59

Based on Lemma 6.14 we are able to define holomorphy of the mapping $z \mapsto u(z)$.

lemma 11 **Lemma 6.15.** *For $z \in \mathcal{A}_\delta$, the function $z \mapsto u(z)$ admits a complex derivative $\partial_{z_j} u(z) \in V$, s.t., for all $v \in V$ and $z \in \mathcal{A}_\delta$,*

$$\int_D a(x,z)\nabla \partial_{z_j} u(x,u) \cdot \nabla v(x) dx = L_0(x) := -\int_D \psi_j(x) \nabla u(x,z) \cdot \nabla v(x) dx. \tag{6.29}$$ 60

Proof. Fix $j \geq 1$ and $z \in \mathcal{A}_\delta$. For $h \in \mathbb{C}\backslash\{0\}$, set

$$w_h(z) := \frac{u(z + he_j) - u(z)}{h} \in V. \tag{6.30}$$ 64

Note that w_h is the unique solution of the variational problem

$$\int_D a(x,z)\nabla w_h(x,z) \cdot \nabla v(x) dx = -\int_D \psi_j \nabla u(x, z + he_j) - \nabla v(x) dx =: L_h(v). \tag{6.31}$$

For $|h|\|\psi_j\|_{L^\infty(D)} \leq \frac{\delta}{2}$, w_h is well-defined, since

$$\frac{\delta}{2} \leq \mathrm{Re}(a(x, z + he_j)) \leq |a(x, z + he_j)| \leq 2R + \frac{\delta}{2}, \quad x \in D \tag{6.32}$$

by stability estimate (6.28)

$$\|u(z+he_j)-u(z)\|_V = \|u(z+he_j)-\nabla u(z)\|_{L^2(D)} \leq |h|\|\psi_j\|_{L^\infty(D)}\frac{4\|f\|_V^*}{\delta} \tag{6.33}$$

and L_h converges towards L_0 in V^* as $h \to \infty$. This implies that w_h converges in V towards a limit $w_0 \in V$ which is the solution of

$$\int_D a(x, z)\nabla w_=(x, z) \cdot \nabla v(x)dx = L_0(v) \tag{6.34}$$

for $v \in V$. Hence, $\partial_q u(z) = w_0$ exists in V and is the unique solution of (6.29). ☐

Note that the analyticity domain $\mathcal{A}_\gamma$ contains polydiscs, i.e. with a sequence $\varrho = (\varrho_j)_{j\geq1}$ of positive radii, define the polydiscs

$$\mathcal{U}_\varrho := \prod_{j\geq1}\{z_j \in \mathbb{C}\colon |z_j| \leq \varrho_j\} = \{z_j \in \mathbb{C}\colon z = (z_j)_{j\geq1}\ ;\ |z_j| \leq \varrho_j\} \tag{6.35}$$ 62

Definition 6.16. A sequence $\varrho = (\varrho_j)_{j\geq1}$ is called *admissible* if, for $x \in D$

$$\sum_{j\geq1} \varrho_j|\psi_j(x)| \leq \mathrm{Re}(a_0(x)) - \delta, \tag{6.36}$$ 63

and $U_\varrho \subset \mathcal{A}_\delta$ for admissible ϱ. In this case, the linear UEAC is equivalent to

$$\sum_{j\geq1} |\psi_j(x)| \leq \mathrm{Re}(a_0(x)) - r, \quad x \in D. \tag{6.37}$$

Example 6.17. *The sequence $\varrho_j = 1$ is δ-admissible for all $0 < \delta \leq r$ and for $\delta < r$ there exists a δ-admissible sequence, s.t. $\varrho_j > 1$ for all $j \geq 1$, i.e. the polydiscs $\mathcal{U}_\varrho$ is strictly larger than $\mathcal{U}$ in every variable.*

lemma 12 **Lemma 6.18** (Estimate of Taylor coefficients)**.** *If UEAC(r,R) holds for some $0 < r \leq R < \infty$ and if $\varrho = (\varrho_j)_{j\geq1}$ is δ-admissible for some $0 < \delta < r$, then, for any $\nu \in \mathcal{F}$, it holds,*

$$\|t_\nu\|_V \leq \frac{\|f\|_{V^*}}{\delta}\prod_{j\geq1}\varrho_j^{-\nu_j} = \frac{\|f\|_{V^*}}{\delta}\varrho^{-\nu}, \tag{6.38}$$ 65

with $t^{-0} = 1$ for $t \geq 0$.

sketch of proof for Theorem 6.13. We assume UEAC(r,R) and analyticity of the mapping $z \mapsto u(z)$, on the domain $\mathcal{A}_\delta$. The proof follows by

(a) a choice of a $\frac{\delta}{2}$-admissible sequence ϱ

(b) establish $\ell^p(\mathcal{F})$-summability of the Taylor coefficients.

By Lemma 6.18 with $\delta = \frac{r}{2}$ follows for the Taylor coefficients

$$\|t_\nu\|_V \leq \frac{2\|f\|_{V^*}}{r} e^{-\varrho} \tag{6.39}$$ 66

(one possible) construction of a δ-admissible vector ϱ:

- select $J_0 \in \mathbb{N}$, s.t. $\sum_{j>J_0} \|\psi_j\|_{L^\infty(D)} \leq \frac{r}{12}$
- partitioning $\mathbb{N}$ in $E := \{1, \dots, J_0\}$ and $F := \mathbb{N}$ E
- choose $\kappa > 1$ s.t. $(\kappa - 1) \sum_{j \leq J_0} \|\psi_j\|_{L^\infty(D)} \leq \frac{r}{4}$
- For each $\nu \in \mathcal{F}$, select $\varrho = \varrho(\nu)$ by

$$\varrho_j := \begin{cases} \kappa & , j \in E \\ \max\{1, \frac{r\nu_j}{4|\nu_F| \|\psi_j\|_{L^\infty(D)}}\} & , j \in F \end{cases} \tag{6.40}$$

  where ν_F denotes the restriction of ν to the set F and $|\nu_F| := \sum_{j \geq J_0} \nu_j$.

- equation (6.38) takes the form

$$\|t_\nu\|_V \leq \frac{2\|f\|_{V^*}}{r} \left(\prod_{j \in E} \mu^{\nu_j} \right) \left(\prod_{j \in F} \left(\frac{|\nu_F| d_j}{\nu_j} \right)^{\nu_j} \right), \tag{6.41}$$ 68

  where $\mu := \frac{1}{\kappa} < 1$ and $d_j := \frac{4\|\psi_j\|_{L^\infty(D)}}{r}$

- $\ell^p(\mathcal{F})$-summability of t_ν: (6.41) has the general form

$$\|t_\nu\|_V \leq C_r \alpha(\nu_E) \beta(\nu_F) \tag{6.42}$$ 69

- Let $\mathcal{F}_E$ be the collection of $\nu \in \mathcal{F}$ with support on E. For $0 < p < \infty$, it holds

$$\sum_{\nu \in \mathcal{F}} \|t_\nu\|_V^p \leq C_r^p \sum_{\nu \in \mathcal{F}} \alpha(\nu_E)^p \beta(\nu_F)^p = C_r^p A_E A_F, \tag{6.43}$$ 70

  with

$$A_E := \sum_{\nu \in \mathcal{F}_E} \alpha(\nu)^p \quad \text{and} \quad A_F := \sum_{\nu \in \mathcal{F}_F} \beta(\nu)^p \tag{6.44}$$

- From $< \infty$ bounds for A_E and A_F follows $\ell^p(\mathcal{F})$-summability of t_ν
- The best N-term convergence rate (6.25) in Theorem 6.13 follows from Stechkin's Lemma.

□

6.1.2 Convergence rates of Legendre expansion

Previously we have seen the rate of convergence for the best N-term truncation of the Taylor expansion in y. Analogously we can give a result on a Legendre expansion. Therefore, we consider two kinds of normalizations. One in L^2 and the other point-wise.

$$\|P_n\|_{L^\infty([-1,1])} = P_n(1) = 1, \tag{6.45}$$ L inf

$$\frac{1}{2}\int_{-1}^{1} |L_n(t)|^2 dt = 1 \tag{6.46}$$ L 2

with $L_n(t) = \sqrt{2n-1}P_n(t)$.

Remark 6.19. Note that since $u \in L^\infty(\Gamma,\mu;V) \subset L^2(\Gamma,\mu;V)$, it admits a unique expansion

$$u(y) = \sum_{\nu\in\mathcal{F}} u_\nu P_\nu(y) = \sum_{\nu\in\mathcal{F}} v_\nu L_\nu(y) \tag{6.47}$$ 71

which converges in $L^2(\Gamma,\mu;V)$ with coefficients $u_\nu, v_\nu \in V$,

$$u_\nu ;= \int_\Gamma u(y)L_\nu(y)\mu(dy) \quad \text{and} \quad v_\nu := \left(\prod_{j\geq 1}(1+2\nu_j)\right)^{\frac{1}{2}} u_\nu. \tag{6.48}$$ 72

Thus,

$$\|v_\nu\|_V \leq \|u\|_V \tag{6.49}$$

since

$$\|u_\nu\|_V = \left(\prod_{j\geq 1}(1+2\nu_j)\right)^{\frac{1}{2}} \|v_\nu\|_V, \quad \nu \in \mathcal{F} \tag{6.50}$$

and therefore it is sufficient to show ℓ^p-summability of $(\|u_\nu\|_V)_{\nu\in\mathcal{F}}$.

lemma 13 **Lemma 6.20.** *Assume UEAC(r,R) for some $0 < r \leq R < \infty$ and a δ-admissible sequence $\varrho = (\varrho_j)_{j\geq 1}$, for some $0 < \delta < r$, that satisfies $\varrho_j > 1$ for all $j \in \mathbb{N}$, s.t. $\nu_j \neq 0$. Then, for any $\nu \in \mathcal{F}$,*

$$\|u_\nu\|_V \leq \frac{\|f\|_{V^*}}{\delta} \prod_{j\geq 1, \nu_j\neq 0} \varphi(\varrho_j)(2\nu_j+1)\varrho_j^{-\nu_j}, \tag{6.51}$$ 73

where $\varphi(t) := \frac{\pi t}{2(t-1)}$ for $t > 1$.

As a consequence we can formulate

theorem 6 **Theorem 6.21.** *For $a(x,z)$ satisfying UEAC(r,R) for some $0 < r \leq R < 0$ and $\left(\psi_j\|_{L^\infty(D)}\right)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p < 1$, the sequences $(\|u_\nu\|_V)_{\nu\in\mathcal{F}}$ and $(\|v_\nu\|_V)_{\nu\in\mathcal{F}}$ belong to $\ell^p(\mathcal{F})$ for some value of p. The Legendre expansion* (6.47) *converges in $L^\infty(\Gamma,\mu;V)$ in the following sense.*

If $(\Lambda_N)_{N\geq 1}$ is any sequence exhausting $\mathcal{F}$, then the partial sums

$$S_{\Lambda_N}u(y) := \sum_{\nu\in\Lambda_N} u_\nu(x)P_\nu(y) = \sum_{\nu\in\Lambda_N} v_\nu(x)L_\nu(y) \tag{6.52}$$

satisfy

$$\lim_{N\to\infty} \sup_{y\in\Gamma} \|u(y) - S_{\Lambda_N} u(y)\|_V = 0. \tag{6.53}$$

If Λ_N is a set of $\nu \in \mathcal{F}$ corresponding to indices of N maximal $\|u_\nu\|_V$,

$$\sup_{y\in\Gamma} \|u(y) - S_{\Lambda_N} u(y)\|_V \le (\|u_\nu\|_V)_{\ell^p(\mathcal{F})} N^{-s}, \quad s := \frac{1}{p} - 1. \tag{6.54}$$

If Λ_N is a set of $\nu \in \mathcal{F}$ corresponding to indices of N maximal $\|v_\nu\|_V$,

$$\|u - S_{\Lambda_N} u\|_{L^2(\Gamma,\mu;V)} \le (\|v_\nu\|_V)_{\ell^p(\mathcal{F})} N^{-s}, \quad s := \frac{1}{p} - \frac{1}{2}. \tag{6.55}$$

6.2 Adaptive stochastic Galerkin FEM

Our aim is to propose an algorithm to construct an index-set $\Lambda \subset \mathcal{F}$ and according finite element spaces, such that the Galerkin approximation of the parametric variational problem converges (almost) quasi-optimally.

Remark 6.22. Using the three term recursion of orthogonal polynomials in (3.18) and the polynomial expansion of (6.47), the coefficients $(u_\nu)_{\nu\in\mathcal{F}}$ are determined by

$$A_0 u_\nu + \sum_{m\ge 1} A_m(\beta^m_{\nu_m+1} u_{\nu+\epsilon_m} + \beta^m_{\nu_m} u_{\nu-\epsilon_m}) = f\delta_{0,\nu}, \quad \text{for all } \nu \in \mathcal{F}. \tag{6.56}$$

Definition 6.23. For $\Lambda \subset \mathcal{F}$ define $\mathrm{supp}(\Lambda) \subset \mathbb{N}$ as set of *active dimensions*

$$\mathrm{supp}(\Lambda) := \bigcup_{\nu\in\mathcal{F}} \mathrm{supp}(\nu). \tag{6.57}$$

`activeDimensions`

Define the *boundary* of Λ as the infinite set

$$\partial\Lambda := \{\nu \in \mathcal{F}\backslash\Lambda \colon \exists m \in \mathbb{N} \text{ s.t } \nu - \epsilon_m \in \Lambda \text{ or } \nu + \epsilon_m \in \Lambda\}. \tag{6.58}$$

`boundary`

Define the *active boundary* of Λ

$$\partial^\circ\Lambda := \{\nu \in \mathcal{F}\backslash\Lambda \colon \exists m \in \mathrm{supp}(\Lambda) \text{ s.t } \nu - \epsilon_m \in \Lambda \text{ or } \nu + \epsilon_m \in \Lambda\}. \tag{6.59}$$

`activeBoundary`

The first part of the subspace construction is based on the chosen active dimension. For $\Lambda \subset \mathcal{F}$ the Galerkin projection of u onto the space

$$\mathcal{V}(\Lambda) := \{v_\Lambda(x,y) = \sum_{\nu\in\Lambda} v_{\Lambda,\nu}(x) P_\nu(y) \colon v_{\Lambda,\nu} \in V\} \subset L^2(\Gamma,\mu;V) \tag{6.60}$$

is the unique $u_\Lambda \in \mathcal{V}(\Lambda)$ satisfying

$$\int_\Gamma \langle A(y)u_\Lambda(y), v_\Lambda(y)\rangle d\mu(y) = \int_\Gamma \int_D f(x) v_\Lambda(y) dx d\mu, \quad \forall v_\Lambda \in \mathcal{V}(\Lambda). \tag{6.61}$$

`LambdaVariational`

To further restrict the problem in (6.61) we assume a conform mesh triangulation $\mathcal{T}$ of D and consider the space of continuous piecewise polynomials of some fixed degree p: $\mathcal{V}_p(\mathcal{T})$. For $\Lambda \subset \mathcal{F}$ and $p \ge 0$ define the subspace of $\mathcal{V}(\Lambda)$

$$\mathcal{V}_p(\Lambda,\mathcal{T}) := \{v_N(x,y) = \sum_{\nu\in\Lambda} v_{N,\nu}(x) P_\nu(y) \colon v_{N,\nu} \in \mathcal{V}_p(\mathcal{T})\} \tag{6.62}$$

Remark 6.24. $\mathcal{V}_p(\mathcal{T})$ is a finite dimensional subspace of $L^2(\Gamma, \mu; V)$ and analogously to (6.61), the Galerkin approximation of u_N of u is the solution of the variational problem

$$\int_\Gamma \langle A(y)u_N(y), v_N(y)\rangle d\mu(y) = \int_\Gamma \int_D f(x)v_N(y)dxd\mu, \quad \forall v_N \in \mathcal{V}_p(\Lambda, \mathcal{T}). \tag{6.63}$$

TVariational

The object of main interest is the residual

Definition 6.25. For any approximation $w_\Lambda \in \mathcal{V}(\Lambda)$ of u the *residual operator* $\mathcal{R}(w_\Lambda) \in L^2(\Gamma, \mu; V^*)$ is given by

$$\mathcal{R}(w_\Lambda) := f - \mathcal{A}w_\Lambda = \mathcal{A}(u - w_\Lambda). \tag{6.64}$$

residualOperator

Given the basis representation we have $\mathcal{R}(w_\Lambda) = \sum_{\nu\in\mathcal{F}} r_\nu(w_\Lambda)P_\nu$ with convergence in $L^2(\Gamma, \mu; V^*)$ and the coefficients are given by

$$r_\nu(w_\Lambda) = f\delta_{\nu,0} - A_0 w_{\Lambda,\nu} - \sum_{m\geq 1} A_m(\beta^m_{\nu_m+1} w_{\Lambda,\nu+\epsilon_m} + \beta^m_{\nu_m} w_{\Lambda,\nu-\epsilon_m}), \quad \nu \in \mathcal{F}, \tag{6.65}$$

i.e.

$$\langle r_\nu(w_\Lambda), v\rangle = \int_D f\delta_{\nu,0} - \sigma_\nu(w_\Lambda)\cdot \nabla v dx \quad \forall\, v \in V \tag{6.66}$$

for

$$\sigma_\nu(w_\Lambda) := a_0\nabla w_{\Lambda,\nu} + \sum_{m\geq 1} a_m \nabla(\beta^m_{\nu_m+1} w_{\Lambda,\nu+\epsilon_m} + \beta^m_{\nu_m} w_{\Lambda,\nu-\epsilon_m}), \quad \nu \in \mathcal{F}. \tag{6.67}$$

Remark 6.26. Note that, $r_\nu(w_\Lambda)$ is nonzero only for $\nu \in \Lambda \cup \partial\Lambda$. Thus, we have the decomposition of the residual on the active domain and the boundary, i.e. $\mathcal{R}(w_\Lambda) = \mathcal{R}_\Lambda(w_\Lambda) + \mathcal{R}_{\partial\Lambda}(w_\Lambda)$ where, for a set $\Xi \subset \mathcal{F}$, we define

$$R_\Xi(w_\Lambda) := \sum_{\nu\in\Xi} r_\nu(w_\Lambda)P_\nu. \tag{6.68}$$

Consequently we obtain

$$\|\mathcal{R}(w_\Lambda)\|^2_{L^2(\Gamma,\mu,V^*)} = \|\mathcal{R}_\Lambda(w_\Lambda)\|^2_{L^2(\Gamma,\mu,V^*)} + \|\mathcal{R}_{\partial\Lambda}(w_\Lambda)\|^2_{L^2(\Gamma,\mu,V^*)}. \tag{6.69}$$

lemma 1.1

Lemma 6.27. *For any* $w_\Lambda \in \mathcal{V}(\Lambda)$,

$$\frac{1}{1+\gamma}\left(\|\mathcal{R}_\Lambda(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V)} + \|\mathcal{R}_{\partial\Lambda}(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V)}\right) \tag{6.70}$$

$$\leq \|w_\Lambda - u\|_{\mathcal{A}} \tag{6.71}$$

$$\leq \frac{1}{1-\gamma}\left(\|\mathcal{R}_\Lambda(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V)} + \|\mathcal{R}_{\partial\Lambda}(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V)}\right) \tag{6.72}$$

Proof. By Riesz representation Theorem in $L^2(\Gamma, \mu; V)$,

$$\|u - w_\Lambda\|^2_{\mathcal{A}} = \sup_{v\in L^2(\Gamma,\mu;V)} \frac{|\langle \mathcal{A}(u - w_\Lambda), v\rangle|^2}{\|v\|^2_{\mathcal{A}}} = \sup_{v\in L^2(\Gamma,\mu;V)} \frac{|\langle \mathcal{R}_\Lambda(w_\Lambda), v\rangle|^2}{\|v\|^2_{\mathcal{A}}}. \tag{6.73}$$

RiezHint

Then, by Cauchy-Schwarz and the equivalenz of the energy norm and the $L^2(\Gamma, \mu; V)$-norm, the assertion follows. □

Lemma 6.28. *For any $w_\Lambda \in \mathcal{V}(\Lambda)$, we have*

$$\frac{1}{1-\gamma}\|\mathcal{R}(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V)} \leq \|w_\Lambda - u_\Lambda\|^2_{\mathcal{A}} \leq \frac{1}{1-\gamma}\|\mathcal{R}_\Lambda(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V)} \tag{6.74}$$

Proof. For any $v_\Lambda \in \mathcal{V}(\Lambda) \subset L^2(\Gamma,\mu;V)$

$$\langle \mathcal{A}(u_\Lambda - w_\Lambda), v_\Lambda\rangle = \langle \mathcal{A}(u - w_\Lambda), v_\Lambda\rangle = \langle \mathcal{R}(w_\Lambda), v_\Lambda\rangle\ . \tag{6.75}$$

Then, the proof follows by 6.73 and the same argument as in Lemma 6.27. □

Note that by Galerkin orthogonality,

$$\|u - u_N\|^2_{\mathcal{A}} = \|u - u_\Lambda\|^2_{\mathcal{A}} + \|u_\Lambda - u_N\|^2_{\mathcal{A}}. \tag{6.76}$$

eq:errorGalerkinOrthogon

In the following we will investigate each part of (6.76) for itself.

6.2.1 Tail estimator

The adpative process will rely on the computation of suitable error estimators for the stochastic *tail* and the physical part of the solution. We first approach the stochastic approximation and define a suitable error indicator yielding a decision support to define the stochastic space approximation, i.e. taking greater polynomial degrees in consideration. For any $w_\Lambda \in \mathcal{V}(\Lambda)$ and $v \in \partial\Lambda$, let

$$\zeta_\nu(w_\Lambda) := \sum_{m=1}^{\infty} \left\|\frac{a_m}{a_0}\right\|_{L^\infty(D)} (\beta^m_{\nu_m+1}\|w_{\Lambda,\nu+\epsilon_m}\|_V + \beta^m_{\nu_m}\|w_{\Lambda,\nu-\epsilon_m}\|_V). \tag{6.77}$$

ZetaSchaetzer

For $\Delta \subset \partial\Lambda$, let

$$\zeta(w_\Lambda, \Delta) := \left(\sum_{\nu\in\Delta} \zeta_\nu(w_\Lambda)^2\right)^{\frac{1}{2}} \tag{6.78}$$

Lemma 6.29. *For any $w_\Lambda \in \mathcal{V}(\Lambda)$*

$$\|\mathcal{R}_{\partial\Lambda}(w_\Lambda)\|_{L^2(\Gamma,\mu;V^*)} \leq \zeta(w_\Lambda, \partial\Lambda). \tag{6.79}$$

Proof. By Parseval's identity we have

$$\|\mathcal{R}_{\partial\Lambda}(w_\Lambda)\|^2_{L^2(\Gamma,\mu;V^*)} = \sum_{\nu\in\partial\Lambda} \|r_\nu(w_\Lambda)\|^2_{V^*}. \tag{6.80}$$

Since $\nu \neq 0$ $(0 \in \Lambda)$, and (64*) and Cauchy Schwarz and triangle inequality leads to

That sentence has to be reformulated

$$\|r_\Lambda(w_\Lambda)\|_{V^*} = \sup \frac{|\langle r_\Lambda(w_\Lambda), v\rangle|}{\|v\|_V} \tag{6.81}$$

$$= \sup \frac{1}{\|v\|_V}\left|\int_D a_0^{-1}\sigma_v(w_\Lambda)\nabla(a_0 v)dx\right| \tag{6.82}$$

$$\leq \|a_0^{-1}v(w_\Lambda)\|_V \leq \zeta_v(w_\Lambda). \tag{6.83}$$

□

Note that $\zeta(w_\Lambda, \partial\Lambda)$ is an infinite sum in (68) due to $|\partial\Lambda| = \infty$. However, for $v \in \partial\Lambda\backslash\partial^\circ\Lambda$, i.e. we have $v = \mu + \epsilon_n$ for $\mu \in \Lambda, m \in \mathbb{N}\backslash\operatorname{supp}(\Lambda)$

$$\zeta_v(w_\Lambda) = \left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} \beta_1^m \|w_{\Lambda,\mu}\|_V. \tag{6.84}$$

zetaSchaetzer2

Summing over all inactive dimensions,

$$\zeta_\mu(w_\Lambda, \Lambda) := \left(\sum_{m\in\mathbb{N}\backslash\operatorname{supp}(\Lambda)} \zeta_{\mu+\epsilon_m}(w_\Lambda)^2 \right)^{\frac{1}{2}} \tag{6.85}$$

zetaSum

$$= \|w_{\Lambda,\mu}\|_V \left(\sum_{m\in\mathbb{N}\backslash\operatorname{supp}(\Lambda)} \left(\left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} \beta_1^m \right)^2 \right)^{\frac{1}{2}} \tag{6.86}$$

for $\mu \in \Lambda$. Hence,

$$\zeta(w_\Lambda, \partial\Lambda)^2 = \sum_{v\in\partial^\circ\Lambda} \zeta_v(w_\Lambda)^2 + \sum_{\mu\in\Lambda} \zeta_\mu(w_\Lambda)^2. \tag{6.87}$$

zetaSquare

6.2.2 Residual based estimation of spatial error

Assume a regular triangulation with a triangle $T \in \mathcal{T}$ of D and edge $E \in \mathcal{E}$, where $\mathcal{E}$ denotes the set of edges for a triangluation, $h_T := \operatorname{diam}(T)$, $h_E := \operatorname{diam}(E)$ and patches $\widetilde{\omega}_T$, $\widetilde{\omega}_E$. Then, there exists some interpolation operator $J\colon H_0^1 \to \mathcal{V}_p(\mathcal{T})$, s.t.

$$\|a_0^{-\frac{1}{2}}(v - Jv)\|_{L^2(T)} \leq c_\mathcal{T} h_T |v|_{V,\widetilde{\omega}_T}, \quad T \in \mathcal{T} \tag{6.88}$$

interpolationOperator

and

$$\|a_0^{-\frac{1}{2}}(v - Jv)\|_{L^2(T)} \leq c_\mathcal{E} h_E^{\frac{1}{2}} |v|_{V,\tilde{\omega}_E}, \quad E \in \mathcal{E}. \tag{6.89}$$

For any $T \in \mathcal{T}$ and $\omega_N \in \mathcal{V}_p(\Lambda, \mathcal{T})$ let,

$$\eta_{\mu,T}(w_N) := h_T \|a_0^{-\frac{1}{2}}(f\delta_{\mu 0} + \nabla\cdot\sigma(w_N))\|_{L^2(T)}, \quad \mu \in \Lambda \tag{6.90}$$

denote the energy in the interior and

$$\eta_{\mu,E}(w_N) := h_E^{\frac{1}{2}} \|a_0^{-\frac{1}{2}} [\![\sigma_\mu(w_N)]\!]\|_{L^2(E)}, \quad \mu \in \Lambda \tag{6.91}$$

the flow over the edge $E = T_1 \cup T_2$, With the jump term

$$[\![\sigma]\!] := \sigma|_{T_1} \times n_1 + \sigma|_{T_2} \times n_2 \tag{6.92}$$

defined by the corresponding normal directions n_i of T_i. Then the physical error indicator is given by

$$\eta_\mu(w_N) := \left(\sum_{T\in\mathcal{T}} \eta_{\mu,T}(w_N)^2 + \sum_{E\in\mathcal{E}} \eta_{\mu,E}(\omega_N)^2 \right)^{\frac{1}{2}}. \tag{6.93}$$

Theorem 6.30. *For $w_N \in \mathcal{V}_p(\Lambda; \mathcal{T})$, $\nu \in \Lambda, v \in V$*

$$\|\langle r_\nu(w_N), v - Jv\rangle\| \leq c_\eta \eta_\nu(w_N) \|v\|_V \tag{6.94}$$

etaSchaetzerTheorem

with $c_\eta > 0$ depending only on a_0 and $\mathcal{T}$.

Proof. Set $z := v - Jv$, $\sigma_\nu := \sigma_\nu(w_N)$. Then, it follows

$$\langle r_\nu(w_N), z\rangle = \sum_{T\in\mathcal{T}} \int_T f\delta_{\nu 0} z - \sigma_\mu \cdot \nabla z dx \tag{6.95}$$

$$= \sum_{T\in\mathcal{T}} \int_T (f\delta_{\nu 0} + \nabla\sigma_\nu) z dx - \sum_{E\in\mathcal{E}} \int_E [\![\sigma_\nu]\!] z ds. \tag{6.96}$$

By Cauchy Schwarz we obtain

$$|\langle r_\nu(w_N), z\rangle| \leq \sum_{T\in\mathcal{T}} \|a_0^{-\frac{1}{2}}(f\delta_{\nu 0} + \nabla\sigma_\nu)\|_{L^2(T)} + \sum_{E\in\mathcal{E}} \|a_0^{-\frac{1}{2}} [\![\sigma_\nu]\!]\|_{L^2(E)} \|a_0^{-\frac{1}{2}} z\|_{L^2(T)}. \tag{6.97}$$

By (6.88)

$$|\langle r_\nu(w_N), z\rangle| \leq c_T \sum_{T\in\mathcal{T}} h_T \|a_0^{-\frac{1}{2}}(f\delta_{0\nu} + \nabla\sigma_\nu)\|_{L^2(T)} |v|_{V,\widetilde{\omega}_\mathcal{T}} \tag{6.98}$$

$$+ c_\mathcal{E} \sum_{E\in\mathcal{E}} g_E^{-\frac{1}{2}} \|a_0^{-\frac{1}{2}} [\![\sigma_\nu]\!]\|_{L^2(E)} |v|_{V,\widetilde{\omega}_E} \tag{6.99}$$

$$\leq c_\eta \eta_\nu(w_N) \|v\|_V. \tag{6.100}$$

□

6.2.3 Upper Bound of total error

Assume some

$$Q\colon L^2(\Gamma,\mu;\mathcal{T}) \to \mathcal{V}_p(\Lambda;\Gamma),\ v \mapsto Qv := \sum_{\nu\in\Lambda} (Iv_\nu) P_\nu \tag{6.101}$$

with

$$v = \sum_{\nu\in\mathcal{F}} v_\nu P_\nu \in L^2(\Gamma,\mu;V), \tag{6.102}$$

and define

$$c_Q := \|id - Q\|_{\mathcal{A}^*}. \tag{6.103}$$

theorem GalerkinError

Theorem 6.31. *For $w_N \in \mathcal{V}_p(\Gamma;\mathcal{T})$ and u_N as the Galerkin projection of u onto $\mathcal{V}_p(\Lambda;\mathcal{T})$, then*

$$\|w_N - u\|_\mathcal{A}^2 \leq \left(\frac{1}{\sqrt{1-\gamma}} \sup_{v\in L^2_\pi(\Gamma;V)} \frac{|\langle \mathcal{R}(w_N), v - Qv\rangle|}{\|v\|_{L^2_\pi(\Gamma;V)}} + c_Q \|w_N - u_N\|_\mathcal{A} \right)^2 \tag{6.104}$$

$$+ \|w_N - u_N\|_\mathcal{A}^2. \tag{6.105}$$

Proof. By orthogonality, we have

$$\|w_N - u\|_\mathcal{A}^2 = \|u_N - u\|_\mathcal{A}^2 + \|w_N - u_N\|_\mathcal{A}^2 \tag{6.106}$$

and

$$\|u_N - u\|_\mathcal{A} = \sup_{v\in L^2(\Gamma,\mu;V)} \frac{|\langle \mathcal{R}(u_N), v\rangle|}{\|v\|_\mathcal{A}} \tag{6.107}$$

$$= \sup_{v\in L^2(\Gamma,\mu;V)} \inf_{v_N\in\mathcal{V}_p(\Gamma;\mathcal{F})} \frac{|\langle \mathcal{R}(u_N), v - v_N\rangle|}{\|v\|_\mathcal{A}}. \tag{6.108}$$

By Cauchy Schwarz we get

$$|\langle \mathcal{R}(u_N) - \mathcal{R}(w_N), v - v_N\rangle| = |\langle \mathcal{A}(w_N - u_N), v - v_N\rangle| \tag{6.109}$$
$$\leq \|w_N - u_N\|_{\mathcal{A}} \|v - v_N\|_{\mathcal{A}}. \tag{6.110}$$

The Claim follows with the norm equivalence and $v_N := Qv$. □

Thm28 **Theorem 6.32.** *For $w_N \in \mathcal{V}_p(\Lambda; \mathcal{T})$ we have,*

$$\|w_N - u\|^2_{\mathcal{A}} \leq \left\{ \frac{c_\eta}{\sqrt{1-\gamma}} \left(\sum_{\nu\in\Lambda} \eta_\nu(w_N)^2 \right)^{\frac{1}{2}} + \frac{c_Q}{\sqrt{1+\gamma}} \zeta(w_N, \partial\Lambda) + c_Q \|w_N - u_N\|^2_{\mathcal{A}} \right\}^2 \tag{6.111}$$
$$+ \|w_N - u_N\|^2_{\mathcal{A}}. \tag{6.112}$$

Proof.

$$\frac{|\langle \mathcal{R}(w_N), v - Qv\rangle|}{\|v\|_{L^2(\Gamma,\mu;V)}} \leq \frac{1}{\|v\|_{L^2(\Gamma,\mu;V}} \underbrace{\|\mathcal{R}_{\partial\Lambda}(w_N)\|_{L^2(\Gamma,\mu;V^*)}}_{\text{Lem. 6.27: } \leq \zeta(w_N, \partial\Lambda} \underbrace{\|v - Qv\|_{L^2(\Gamma,\mu;V)}}_{\leq c_Q \|v\|_{L^2(\Gamma,\mu;V)}} \tag{6.113}$$
$$+ \frac{1}{\|v\|_{L^2(\Gamma,\mu;V)}} \sum_{\nu\in\Lambda} |\langle r_\nu(w_N), [v - Qv]_\nu\rangle| \tag{6.114}$$

For the latter we have by Theorem 6.31

$$\sum_{\nu\in\Lambda} |\langle r_\nu(w_N), [v - Qv]_\nu\rangle| \leq c_\eta \sum_{\nu\in\Lambda} \eta_\nu(w_N) \|v_\nu\| \tag{6.115}$$
$$\leq c_\eta (\sum_{\nu\in\Lambda} \eta_\nu(w_N)^2)^{\frac{1}{2}} (\sum_{\nu\in\Lambda} \|v_\nu\|^2_V)^{\frac{1}{2}}, \tag{6.116}$$

which concludes the statement. □

6.2.4 refinement strategy (edge based FEM)

For $E \in \mathcal{E}$ and for $\nu\Lambda$ define

$$\widehat{\eta}_{\nu,E}(w_N) := \left(\eta_{\nu,E}(w_N)^2 + \frac{1}{d+1} \sum_{T:\ E\in\mathcal{E}\cup\partial T} \eta_{\mu,T}(w_N)^2 \right)^{\frac{1}{2}} \tag{6.117}$$

In the physical domain we use the marking method of Dörfler, i.e., for a parameter $0 \leq \varrho_\eta < 1$, we refine a set of edges $\widehat{\mathcal{E}}_\eta \subset \mathcal{E}$, if

$$\sum_{\nu\in\Lambda} \sum_{R\in\widehat{\mathcal{E}}_\eta} \eta_{\nu,E}(w_N)^2 \geq \varrho_\eta^2 \sum_{\nu\in\Lambda} \eta_\nu(w_N)^2. \tag{6.118}$$

Similar, in the stochastic domain, marking with $0 < \varrho_\zeta < 1$ w.r.t the estimator ζ or for all indices larger than a given threshold, we e.q. add indices to Λ, which are in the following set

$$\left\{ v \in \partial^0\Lambda\colon \zeta_v(w_N) \geq \varrho_\zeta \left(\sum_{v\in\partial^0\Lambda} \zeta_(w_N)^2 \right)^{\frac{1}{2}} \right\} \tag{6.119}$$

and if some of these indices fulfills $v = \nu + \epsilon_m$ with $\nu \in \Lambda$ and $m = \max(\mathrm{supp}(\Lambda))$, then we add an index $v\prime = \nu + \epsilon_m$ with $m\prime = \min(\mathbb{N} \setminus \mathrm{supp}(\Lambda))$ and $\zeta_{v\prime}(w_N) = \left\| \frac{a_m}{a_0} \right\|_{L^\infty(D)} \|w_{N,\mu}\|_V$ is also considered for marking.

The algorithm reads as follows:

1. compute approximation $w_N \approx u_N$ in $\mathcal{V}_p(\Gamma; \mathcal{T})$
2. compute estimators $\eta_{\nu.T}$, $\eta_{\nu,E}$ and ζ_ν
3. consistence error $\|w_N - u_M\|$ can be estimated by residual
4. decide whether to refine mesh $\mathcal{T}$ or increase active set Λ

6.3 Structure of discrete operator

Recall the variational problem representation using the parametrization and the recursive structure of the polynomials. Given a Triangulation $\mathcal{T}$ and an active index set Λ, for $v \in \mathcal{V}_p(\mathcal{T}, \Lambda)$ we have

$$\langle A_0 u_{N,\nu}, v\rangle + \sum_{m\geq 1} \left(\langle \beta^m_{\nu_m+a} A_m u_{N,\nu_m+\epsilon_m}, v\rangle + \langle \beta^m_{\nu_m} A_m u_{N,\nu_m-\epsilon_m}, v\rangle\right) = \langle f\delta_{\nu_0}, v\rangle. \tag{6.120}$$

Now, let $N := \dim \mathcal{V}_p(\mathcal{T}, \Lambda) < \infty$, $\tilde{N} := |\Lambda| N$. The operator equation $\mathbb{A}\mathbf{u} = f$ can be written as

$$\begin{bmatrix} \ddots & & & & & & \ddots \\ \dots & A_0 & \dots & C_m & \dots & 0 & \dots \\ \dots & B_m & \dots & A_0 & \dots & C_m & \dots \\ \dots & 0 & \dots & B_m & \dots & A_0 & \dots \end{bmatrix} \begin{bmatrix} \dots \\ u_{\nu-\epsilon_m} \\ \dots \\ u_\nu \\ \dots \\ u_{\nu+\epsilon_m} \\ \dots \end{bmatrix} = \begin{bmatrix} \dots \\ f_{\nu-\epsilon_m} \\ \dots \\ f_\nu \\ \dots \\ f_{\nu+\epsilon_m} \\ \dots \end{bmatrix} \tag{6.121}$$

with symmetric $\mathbb{A} \in \mathbb{R}^{\tilde{N},\tilde{N}}$, $\mathbf{u} \in \mathbb{R}^{\tilde{N}}$ and for $\mathcal{V}_p(\mathcal{T}, \Lambda) = \mathrm{span}\{\varphi_i\}_{i=1,\dots N}$ we have

$$[A_0]_{i,j} := \langle A_0 \varphi_i, \varphi_j\rangle . [B_m]_{i,j} := \beta^m_{\nu_m} \langle A_m \varphi_i, \varphi_j\rangle [C_m]_{i,j} := \beta^m_{\nu_m+1} \langle A_m \varphi_i, \varphi_j\rangle \tag{6.122}$$

$A_m, B_m, C_m \in \mathbb{R}^{N,N}$ and

$$[B_m]_{i,j} := \beta^m_{\nu_m} \langle A_m \varphi_i, \varphi_j\rangle [C_m]_{i,j} := \beta^m_{\nu_m+1} \langle A_m \varphi_i, \varphi_j\rangle \tag{6.123}$$

7 Tensor Representation of the Model Problem

Let $V := \mathcal{X} \otimes \mathcal{Y}$ with $\mathcal{X} := H^1_0(D)$, $\mathcal{Y} := \bigotimes_{m=1}^\infty L^2(\Gamma_m, \mu_m)$ and $V_M := \mathcal{X} \otimes \mathcal{Y}_M$ with $\mathcal{Y}_M := \bigotimes_{m=1}^M L^2(\Gamma_m, \mu_m)$. Moreover, consider the full tensor set with dimensions d_m, i.e.

$$\Lambda := \{(\mu_1, \dots, \mu_M, 0, \dots) \in \mathcal{F} : \mu_m = 0, \dots, d_m - 1; m = 1, \dots, M\} \tag{7.1}$$

with cardinality $|\Lambda| = \prod_{m=1}^M d_m$ which grows exponentially for $d_m = 1$. Let

$$V(\Lambda) := \mathcal{X} \otimes \mathcal{Y}(\Lambda) := \mathcal{X} \otimes (\bigotimes_{m=1}^M \mathcal{Y}_m) \tag{7.2}$$

with $\mathcal{Y}_m := \text{span}\{P_{\mu_m} : \mu_m = 0, \dots, d_m - 1\}$. The tensor structure of V leads to a formulation in tensor representation. For any $u_N \in \mathcal{V}_p(\Gamma; \Lambda)$,

$$u_N(x, y) = \sum_{i=0}^{N-1} \sum_{\nu \in \Lambda} U[i, \nu]\varphi_i(x)P_\nu(y). \tag{7.3}$$

Set

$$K_m(i, j) := \int_D a_m(x)\nabla\varphi_i(x)\dot{\varpi}_j(x)dx \tag{7.4}$$

for $i, j = 0, \dots, N - 1$ and

$$B_m(\nu, \nu') := \int_{\Gamma_m} P_{\nu_m}(y_m)P_{\nu'_m}(y_m)d\mu_m(y_m) \tag{7.5}$$

for $\nu, \nu' \in \mathcal{F}$ and $m = 0, \dots, M$ with $y_0 = 1$. Then, the model problem can be written as

$$\mathbb{A}(U) := \left(\sum_{m=1}^{M} \mathbb{A}_m\right)(U) = \mathbb{F} \tag{7.6}$$

with

$$\mathbb{A}_m := K_m \otimes I \otimes \cdots \otimes B_m \otimes \cdots \otimes I \tag{7.7}$$

$$\mathbb{F} := f \otimes e_1 \otimes \cdots \otimes e_1 \text{ .} \tag{7.8}$$

Part III
Inverse Problems

8 General Problem Structure

Our aim is to gain knowledge about an unknown system input datum u taking values in some polish space $(X, \|\cdot\|_X)$ from some given model observations δ in another polish space $(Y, \|\cdot\|_Y)$ often isomorphically to some finite real vector space. Usually, the unknown represents a system parameter, e.q. a diffusion coefficient, a boundary condition or some geometric properties. The described relation yields the equation

$$\delta = (\mathcal{O} \circ G)(u). \tag{8.1}$$

eq:inverseEquation

The *input to solution* or *forward* operator $G \colon X \to V$ maps the input parameter onto the solution of the mathematical model. E.q. the permeability coefficient of a soil layer setting onto the hydraulic head of the ground water flow. Whereas, the *observation* operator $\mathcal{O} \colon V \to Y$ yields the measurement, e.q. at specified sensors. Typically, those class of problems is ill-posed. Either due the lack of a unique solution in general or the a sensitive dependence on the measurements. Classical approaches try to find a solution of (8.1) by optimization of e.q. a least squares functional

$$\text{argmin}_{u \in V} \frac{1}{2}\|y - (\mathcal{O} \circ G)(u)\|_Y^2. \tag{8.2}$$

eq:leastSquaresOptim

Just scratching the topic, solving the above minimization can be difficult. The Functional does not need to have a global minimum or minimizing sequences may have no limit in X. By introducing suitable and feasible regularisations, one can solve the modified system

$$\operatorname{argmin}_{u\in E}\left(\frac{1}{2}\|y-(\mathcal{O}\circ G)(u)\|_{Y}^{2}.+\frac{1}{2}\|u-m_0\|_{E}^{2}\right) \tag{8.3}$$

eq:leasSquaresRegularize

for a restricting domain $E\subset X$ and a value $m_0\in E$. However, the choice of the control term and the space E and m_0 respectively is somewhat arbitrary.

We will introduce a statistical approach that desires to determine the distribution of the unknown datum given the knowledge from the measurements. In contrast to the deterministic approach, we can show well-posedness of the probabilistic formulation using Bayes Theorem and we will even present an à priori convergence analysis for the polynomial chaos approximation.

8.1 Bayesian Inversion

We aim to find a *posterior* measure μ_δ on X that contains the information of $u\in X$, taking the system responses $\delta\in Y\cong\mathbb{R}^K$ in account. Moreover, we assume there is à priory knowledge about u, modeled in a *prior* measure μ_0. By introducing a prior measure, different to a uniform distribution, we add an additional regularization into the model, analogously to (8.3). The probabilistic formulation allows to assume a model error

$$\delta=(\mathcal{O}\circ G)(u)+\eta,\quad \eta\sim Q_0. \tag{8.4}$$

eq:InverseEquationNoise

This random variable with values in Y contains errors due to measurements inaccuracy and model mismatch as well. We call η the *observational noise.*

The random variable $\delta|u$ (δ given u) is a conditional random variable, whose distribution Q_u is the translation of Q_0 by $(\mathcal{O}\circ G)(u)$. Assuming $Q_u\langle\langle Q_0$ for $u\in X$ μ_0-a.s., there exists a functional $\Phi\colon X\times Y\to\mathbb{R}$, such that the Radon-Nikodym derivative is given by

$$\frac{dQ_u}{dQ_0}(\delta)=\exp(-\Phi(u,\delta)). \tag{8.5}$$

eq:RNDerivErrorMeasure

Assuming measureability of Φ with respect to the joint (probably non product) measure $\nu_0:=Q_0\otimes\mu_0$, then the joint random variable $(u,\delta)\in X\times Y$ is distributed according to some $\nu\langle\langle\nu_0$ with

$$\frac{d\nu}{d\nu_0}(u,\delta)=\exp(-\Phi(u,\delta)). \tag{8.6}$$

eq:RNDerivProductMeasure

Finally, using the specific choice of measurements $\delta\in Y$ we obtain Bayes Theorem.

Theorem 8.1. *Let $\Phi\colon X\times Y\to\mathbb{R}$ be ν_0 measureable and for $\delta\in Y$, Q_0-a.s.*

$$Z:=\mathbb{E}_{\mu_0}\left[\exp\left(-\Phi(u,\delta)\right)\right]=\int_X e^{-\Phi(u,\delta)}d\mu_0(u)>0. \tag{8.7}$$

eq:ZPositive

Then, the distribution of $u|\delta\sim\mu_\delta$ exists under ν and is absolutely continuous w.r.t μ_0. Furthermore, the Radon-Nikodym derivative is given by

$$\frac{d\mu_u}{d\mu_0}(u)=\exp(-\Phi(u,\delta))\quad \nu\text{-a.s.} \tag{8.8}$$

eq:RNDBayes

Remark 8.2. Denote the density of the noise measure by $Q_0(d\delta) = \varrho(\delta)$. Then,

$$Q_u(\delta) = \varrho(\delta - (\mathcal{O} \circ G)(u)) \tag{8.9}$$

eq:TranslatedErrorDensit

and the *Bayesian potential*

$$\Phi(u, \delta) = -\log\ \varrho(\delta - (\mathcal{O} \circ G)(u)) \tag{8.10}$$

eq:logLikelihood

is the *negative log likelihood*.

Example 8.3. *A widely accepted choice for the noise measure is a Gaussian measure $Q_0 = \mathcal{N}(0, \Gamma)$, for some symmetric positive definite covariance operator $\Gamma\colon Y \times Y \to Y$. Then, the potential takes the form of the least squares function in* (8.2) *measured in a norm induced by the noise covariance*

$$\Phi(u, \delta) = \frac{1}{2}\|\delta - (\mathcal{O} \circ G)(u)\|_\Gamma^2 := \frac{1}{2}\|\Gamma^{-\frac{1}{2}}(y - (\mathcal{O} \circ G)(u))\|_Y. \tag{8.11}$$

eq:BayesPotential

In the following the want to get rid of the assumptions in Bayes theorem and replace them with natural assumptions on our model Therefore, we introduce the following assumptions, containing modeling statements and assumptions on the input to observation operator.

ForwardAssumption

Assumption 8.4.

(i) The space of measurements is identified with $\mathbb{R}^K$ for some $K \in \mathbb{N}$.

(ii) The measurement operator is a vector of bounded linear functionals $\mathcal{O} = (o_1, \dots, o_K)$.

(iii) The input to measurement operator $(\mathcal{O} \circ G)\colon X \to \mathbb{R}^K$ is bounded in the sense that for any $\epsilon > 0$ exists an $M > 0$, such that for all $u \in X$

$$|(\mathcal{O} \circ G)(u)|_\Gamma \leq \exp\left(\epsilon\|u\|_X^2 + M\right). \tag{8.12}$$

eq:boundedMeasurement

Here, $|\cdot|_\Gamma$ denotes the norm induced by the usual euclidean inner product weighted by Γ, i.e.

$$|u|_\Gamma^2 = \langle v, v\rangle_\Gamma := \langle v, \Gamma^{-\frac{1}{2}}v\rangle. \tag{8.13}$$

eq:EuclideanIPGamma

(iv) The operator $(\mathcal{O} \circ G)$ is Lipschitz continuous, i.e. for all $r > 0$ there exists a $K > 0$, such that for all $u_1, u_2 \in X$ it holds

$$\|(\mathcal{O} \circ G)(u_1) - (\mathcal{O} \circ G)(u_2)\|_\Gamma \leq K\|u_1 - u_2\|_X. \tag{8.14}$$

eq:LipschitzMeasurement

Measureability Potential

Proposition 8.5. *Given assumption 8.4, the bayesian potential is measureable with respect to ν_0 and $Z > 0$ Q_0-a.s.*

Proof. Measureability follows directly from the continuity of $(\mathcal{O} \circ G)$ and the second statement is a result of (8.12). □

8.2 Parametric Diffusion Problem

Consider the diffusion model problem (4.8) with unknown coefficient field $u \in X$ and solution $q \in V$. Moreover, assume the UEA(r,R) for $0 < r \leq R < \infty$ holds for u.

lemma ForwardLipschitz **Lemma 8.6.** *Let $u_1, u_2 \in X$ with UEA(r,R), yielding solutions $q_1, q_2 \in V$ of the diffusion variational problem for the same r.h.s $f \in V^*$. Then, the forward map $G\colon X \to V$, $u \mapsto q$ is Lipschitz*

$$\|G(u_1) - G(u_2)\|_V = \|q_1 - q_2\|_V \leq \frac{\|f\|_{V^*}}{r^2}\|u_1 - u_2\|_{L^\infty(D)}. \tag{8.15}$$ eq:LipschitsForward

Proof. The result follows analogously to (5.14) applied to the solution difference. □

Using the parametrization assumption 6.6 for the unknown u we can show that the Bayesian potential is holomorphic in some complex domain.

UEAC+ **Assumption 8.7.** *Additional to UEAC(r,R) for some $0 < r \leq R < \infty$ we assume there exists a real number $\kappa \in (0,1)$, such that*

$$\big\|\|\psi_j\|_{L^\infty(D)}\big\|_{\ell^1(\mathbb{N})} \leq \kappa r. \tag{8.16}$$ eq:kappaBound

Proposition 8.8. *Given 8.7 there exists a sequence $(\kappa_j)_{j\geq 1}$ such that $\|\psi_j\|_{L^\infty(D)} \leq \kappa_k r$ and for all $x \in D$ and $y \in [-1,1]^{\mathbb{N}}$ we have for any $m \in \mathbb{N}$*

$$u(x,y) \geq r\left(1 - (\kappa - \kappa_m) - \kappa_m\right) \tag{8.17}$$

$$\geq r\left(1 - (\kappa - \kappa_m)\right)\left(1 - \frac{\kappa_m}{1 - (\kappa - \kappa_m)}\right) \tag{8.18}$$

$$\geq \tilde{r}\left(1 - \tilde{\kappa}_m\right), \tag{8.19}$$

where $\tilde{r} := r(1-\kappa)$ and $\tilde{\kappa}_m := \kappa_m(1 - (\kappa - \kappa_m))^{-1} \in (0,1)$.

lemma potentialHolomorphy **Lemma 8.9.** *Under 8.7 and given measurements $\delta \in \mathbb{R}^K$ the mappings*

$$\Phi(u(x,\cdot),\delta)\colon [-1,1]^{\mathbb{N}} \to \mathbb{R}, \tag{8.20}$$

$$\exp\left(-\Phi(u(x,\cdot),\delta)\right) : [-1,1]^{\mathbb{N}} \to \mathbb{R} \tag{8.21}$$

and

$$\exp\left(-\Phi(u(x,\cdot),\delta)\right) G(u(x,\cdot))\colon [-1,1]^{\mathbb{N}} \to V \tag{8.22}$$ eq:QOI

admits a unique extension to strongly holomorphic functions on the strips

$$S_\varrho := \bigotimes_{j\geq 1}\left\{y_j + iz_j\colon |y_j| y \frac{\varrho_j}{\tilde{\kappa}_j},\ z_j \in \mathbb{R}\right\} \tag{8.23}$$ eq:complexStrips

for any sequence $\varrho = (\varrho_j)_{j\geq 1}$ with $\varrho_j \in (\tilde{\kappa_j}, 1)$.

Proof. See Lemma 4.7. in Schwab, Stuart - Sparse Deterministic Approximation of Bayesian Inverse Problems. □

We have seen that system responses, the Bayesian potential and a special quantity of interest (8.22) depend holomorphically on the parameter $z \in \mathcal{A}_r \subset \mathcal{C}^{\mathbb{N}}$ in the representation

$$u = u_0 + \sum_{j \geq 1} z_j \psi_j. \tag{8.24}$$

Given that, we are able to define bounds on the parametric density expression.

n parametricDensityBound

Theorem 8.10. *Under assumption 8.7 we have for the parametric density expression*

$$\Theta(z) := \exp\left(-\Phi(u,\delta)|_{u=u_0+\sum_{j\geq 1} z_j \psi_j} \right) \tag{8.25}$$

eq:theta

and a $\tilde{r} < r$

$$\sup_{z \in \mathcal{A}_{\tilde{r}}} |\Theta(z)| = \sup_{z \in \mathcal{A}_{\tilde{r}}} \left| \exp\left(-\Phi(u,\delta)|_{u=u_0+\sum_{j\geq 1} z_j \psi_j} \right) \right| \tag{8.26}$$

eq:thetaBound

$$\leq \exp\left(\frac{\|f\|_{V^*}}{\tilde{r}^2} \sum_{k=1}^{K} \|o_k\|_{V^*}^2 \right). \tag{8.27}$$

8.3 Polynomial Chaos Approximation

The parametric posterior density can be expressed in terms of a Legendre basis, analogously to the forward solution

$$\Theta(y) = \sum_{\nu \in \mathcal{F}} \vartheta_\nu P_\nu(y). \tag{8.28}$$

eq:posteriorLegendre

Truncation to some finite dimensional index set $\Lambda_N \subset \mathcal{F}$ with maximal gpc degree $N \in \mathbb{N}$ yields the approximation

$$\Theta_N(y) = \sum_{\nu \in \Lambda} \vartheta_\nu P_\nu(y). \tag{8.29}$$

eq:posteriorLegendreAppr

Lemma 8.11. *Assume, for some* $\sigma \in (0,1]$, $(\vartheta_\nu)_{j\geq 1} \in \ell^\sigma(\mathbb{N})$. *Then,*

$$\|\Theta(y) - \Theta_N(y)\|_{L^2([-1,1]^{\mathbb{N}},\mu_0)} \leq N^{-(\frac{1}{\sigma}-\frac{1}{2})} \|(\vartheta_\nu)\|_{\ell^\sigma(\mathcal{F})}. \tag{8.30}$$

eq:posteriorConvergence

Proof. The argument follows directly from the previous section and the application of the Stechkin Lemma and Parsevall's identity. ☐

Remark 8.12. Taylor expansion yields analogous results for the L^1-norm and point-wise estimates.

8.4 Sparsity of the Posterior

TO obtain the desired rates of convergency $> \frac{1}{2}$, we need the summability of the coefficient series of the Legendre expansion (8.28) and the Taylor expansion

$$\Theta(y) = \sum_{\nu \in \mathcal{F}} \tau_\nu y^\nu \tag{8.31}$$

eq:posteriorTaylor

respectively. This sparsity will be obtained under the following assumption

assumption Sparsity

Assumption 8.13. *The fluctuation of the input is* σ*-summable, i.e. there exists a* $\sigma \in (0,1]$, *such that*

$$\left(\|\psi_j\|_{L^\infty(D)} \right)_{j \geq 1} \in \ell^\sigma(\mathbb{N}). \tag{8.32}$$

eq:summability

9 Kernel Ridge Regression

Consider a compact set $D \subset \mathbb{R}^d$ and a continuous kernel

$$k\colon D \times D \to \mathbb{R},\ (x, y) \mapsto K(x, y) \in \mathbb{R} \tag{9.1}$$

which is symmetric, i.e. $k(x, y) = k(y, x)$ for all $x, y \in D$. Moreover, consider the integral operator $K\colon L^2(D) \to L^2(D)$ induced by the kernel, given by

$$Ku(x) := \int_D k(x, y)u(y)dy \tag{9.2}$$

eq:integralOperator

By construction, K is compact and symmetric and we have

$$\langle Ku, v\rangle = \langle u, Kv\rangle \tag{9.3}$$

eq:Kernelsymmetric

where $\langle\cdot,\cdot\rangle$ denotes the standard $L^2(D)$ inner product.

Kernel Assumption 1

Assumption 9.1. *We assume K is positive semi definite, i.e.*

$$\langle Ku, u\rangle \geq 0 \quad \textit{for all } u \in L^2(D). \tag{9.4}$$

Corollary 9.2. *Given assumption 9.1 there exists an complete orthonormal system $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $L^2(D)$ and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ with $\lambda_k \geq 0$ for $k \in \mathbb{N}$, s.t.*

$$Ku(x) = \sum_{k=0}^{\infty} \lambda_k \tilde{\varphi}(x)_k \langle\tilde{\varphi}_k, u\rangle. \tag{9.5}$$

Proof. Follows directly from the compactness of K and its spectral representation. □

Consider the following Regression Problem:
Given sample points $x_i \in \mathbb{R}^d$ and sample values $y_i \in \mathbb{R}$ for $i = 1, \dots, p$, we are interested in a function $f \in L^2(D)$, s.t. $y_i \approx f(x_i)$.

9.1 Interpretation As Minimization Problem

Take a CONS $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ and try to approximate this basis. Therefore, define a *Loss-function*, which is interpreted as the *empirical least square error*

$$\frac{1}{2}\sum_{i=1}^{p} |y_i - f(x_i)|^2, \text{ where } \quad f(x_i) = \sum_{k=1}^{\infty} \tilde{\alpha}_k \tilde{\varphi}_k(x). \tag{9.6}$$

Modification by $\varphi_k = \lambda_k \tilde{\varphi}_k$ and imposing a regularisation (or penalty) term

$$\frac{\lambda}{2}\sum_{k\in\mathbb{N}} |\alpha_k|^2 =: \frac{\lambda}{2}\|\alpha\|_{\ell^2(\mathbb{N})} \tag{9.7}$$

yields to the functional

$$J(\alpha) := \frac{1}{2}\sum_{i=1}^{p} |y_i - \sum_{k\in\mathbb{N}} \alpha_k \varphi_k(x_i)|^2 + \frac{\lambda}{2}\|\alpha\|^2_{\ell^2(\mathbb{N})} \tag{9.8}$$

eq:Functional

which has to be minimized. The problem is, in a deterministic setting, the functional without the regularisation term has in general no unique minimum. On the other hand, the dependence on the *hyperparemter* λ yields a further variable and the correct choice of λ is not trivial and mostly done heuristically via e.q. cross validation. This idea is sometimes referred as *Tychonev regularisation*. Assuming a deterministic noise on the interpolation nodes

$$y_i = f(x_i) + \epsilon_i \tag{9.9}$$

gives the chance to chose λ, such that the error, made by the regularisation, is below this ϵ-noise.

The classical machine learning approach is to find a linear approximation in a transformed *featured* space. Therefore, consider $\operatorname{span}\{\varphi_k : k \in \mathbb{N}\}$ as featured space. Our specific choice is $\varphi_k := \sqrt{\lambda_k}\tilde{\varphi}_k$ as eigenfunction of the integral operator K. Furthermore, define the matrix

$$[\Phi_{k,i}]_{k\in\mathbb{N},i=1,\dots,p} := [\varphi_k(x_i)]_{k\in\mathbb{N},i=1,\dots,p} \in \mathbb{R}^{\infty\times p} \tag{9.10}$$

Then, using $\Phi^T\alpha = \sum_{k\in\mathbb{N}} \alpha_k\varphi_k(x_i)$ we obtain the functional

$$J(\alpha) := \frac{1}{2}\left\langle y - \Phi^T\alpha, y - \Phi^T\alpha\right\rangle + \frac{\lambda}{2}\langle\alpha,\alpha\rangle \tag{9.11}$$

which needs to be minimized. Useful and sufficient is the Gaussian normal equation

$$0 = \langle\nabla J(\alpha), v\rangle = -\langle\Phi y, v\rangle + \frac{1}{2}\left\langle\left(\Phi\Phi^T - 2\lambda I\right)\alpha, v\right\rangle, \quad \forall v \in \mathbb{R}^\infty. \tag{9.12}$$

Therefore, by coefficient evaluation, the infinite equation system has to be solved

$$\frac{1}{2}\left(\Phi\Phi^T - 2\lambda I\right)\alpha = \Phi y, \tag{9.13}$$

with the substituation $\alpha = \Phi\beta$ for $\beta = (\beta_i)_{i=1}^p \in \mathbb{R}^\infty$ and using inectivity of Φ:

$$\frac{1}{2}(\Phi\Phi^T + 2\lambda I)\Phi\beta = \Phi y \tag{9.14}$$

$$\frac{1}{2}\Phi(\Phi^T\Phi + 2\lambda I)\beta = \Phi y \tag{9.15}$$

$$\underbrace{\frac{1}{2}(\Phi^T\Phi + 2\lambda I)}_{\in\mathbb{R}^{\infty\times p}}\beta = y \tag{9.16}$$

$$\beta = 2(\Phi^T\Phi + 2\lambda I)^{-1}y \tag{9.17}$$

whereas, the matrix

$$[\Phi^T\Phi]_{i,j} = \sum_{k\in\mathbb{N}} \varphi_k(x_i)\varphi_k(x_j) = k(x_i,x_j) =: T = (T_{i,j}) \in \mathbb{R}^{p\times p}. \tag{9.18}$$

By that, we do not need the basis functions $(\varphi_k)_{k\in\mathbb{N}}$ to describe the system. For our desired function f then it holds

$$f(x) = \Phi^T(x)\alpha := \sum_{k\in\mathbb{N}} \varphi_k(x)\alpha_k = \sum_{k\in\mathbb{N}} \varphi_k(x)\sum_{i=1}^p \varphi_k(x_i)\beta_i = \sum_{i=1}^p k(x,x_i)\beta_i, \tag{9.19}$$

which again makes no use of the basis $(\varphi_k)_{k\in\mathbb{N}}$ but of the kernel k instead. This trick, by introducing a kernel and getting rid of the basis functions, is called *kernel trick*.

Remark 9.3. Summarizing the important facts

1. The basis functions φ_k are not needed explicit.
2. The dimension of $D \subset \mathbb{R}^d$ enters only weakly, i.e. the curse of dimensionality is resolved.
3. $|\beta| \leq p$, $O(p)$, $|K| = O(p^2)$, the $|$solution $| \cong O(p^3)$, $|f(x)| = O(p^2)$, i.e. the method is practical for moderate $p \cong 10^{3-5}$.
4. The question is now, how to choose k?

Example 9.4. *Let $k(x, y)$ be the correlation kernel and K the correlation operator of the Karhunen Loeve operator obtained from the covariance*

$$c(x, y) = k(x, y) - \int_D k(x, y)dy - \int_D k(x, y)dx. \tag{9.20}$$

Example 9.5. *Let $k(x, y) = \mathcal{K}(|x - y|)$ for $x, y \in \mathbb{R}^d$. In that case, the kernel $\mathcal{K}\colon \mathbb{R} \to \mathbb{R}$ consists of radial basis functions. Examples are the* Metern kernel*, the* Laplace kernel.

Example 9.6. *The standard and most cited kernel is the* Gauss-kernel *for example in $\mathbb{R}^d$.*

$$k(x, y) = e^{-\delta\|x-y\|^2} = \prod_{k=1}^{d} e^{-\delta(x_k - y_k)^2} \tag{9.21}$$

Example 9.7. *Classically, one can consider the polynomial kernel*

$$k(x, y) = \langle x, y\rangle^n \tag{9.22}$$

or even better

$$k(x, y) = (1 + \langle x, y\rangle)^n \tag{9.23}$$

for polynomials of degree n.

Example 9.8. *The wavelet kernel is given by*

$$k(x, y) = \sum_{l,k} \frac{1}{2^{lk}} \Psi_k^l(x)\Psi_k^l(y) \tag{9.24}$$

Remark 9.9. For example consider the Gauss kernel to approximate a function $f(x) = \sum_{i=1}^{p} \beta_i e^{\delta|x_i - y_i|^2}$, using the p-(basis)-functions $x \mapsto k(x_i, x)$ for $i = 1, \ldots p$.

9.1.1 Reproducing Kernel Hilbert Spaces

Fundamental theory depends on the work of *Aronszayn* (1950-60).

Consider the Hilbert space $H \subset L^2(D) \cap C^0(D)$ with inner product $\langle \cdot, \cdot \rangle_H$ and induced norm $\| \cdot \|_H$. This space is related to the operator K in the sense that

$$\langle u, v \rangle_H = \langle u, K^{-1} v \rangle = \sum_{k \in \mathbb{N}} \langle u, \varphi_k \rangle \langle \varphi_k, v \rangle. \tag{9.25}$$

For the Matern kernel,this space is just the Sobolev space. For Gauss kernel, this space is a native function space.

This approach gives raise to the interpretation

$$J(f) := \frac{1}{2} \sum_{i=1}^{p} |y_i - f(x_i)|^2 + \frac{\lambda}{2} \langle f, f \rangle_H \tag{9.26}$$

i.e. we minimize the empirical risk (or the loss functional) under the assumption that our function is element of a native space. Even, this approach is equivalent to the Kernel Ridge regression.

Corollary 9.10. *It holds, for $f \in H$*

$$f(x) = \langle k(x, \cdot), f(\cdot) \rangle_H = \sum_{k \in \mathbb{N}} \varphi_k(x) \langle \varphi_k, f \rangle. \tag{9.27}$$

This property is called Reproducing property*.*

Remark 9.11. The native spaces are dense in $L^2(D)$.

9.2 Interpretation As Bayesian Inverse Problem

An alternative to solve the minimization problem defined by J is to define our desired interpolant function as expectation of a conditional random variable

$$f(x) := \mathbb{E}[(y \mid x)]. \tag{9.28}$$

This can be tackled by the Bayesian formulation, which yields the Radon-Nikodym derivative $p(y \mid x)$ w.r.t to some prior measure. This statistical considers random variables which yields dependence on some unmentioned probability space, captured in the notion of dependence of ξ.

Let $x = x_1, \dots, x_p$ with $x_i \in D \subset \mathbb{R}^d$ sample points with corresponding values $y = y_1, \dots, y_m$ with $y_i \in \mathbb{R}$ for $i = 1, \dots, p$. Our aim is to find a function $\tilde{f}(x) := \mathbb{E}[(x \mid y)] := \int \tilde{f}(x, \xi) p(\xi) d\xi$, where, for a vector $\xi \in \mathbb{R}^\infty$, the function

$$(x, \xi) \mapsto f(x, \xi) := (\varphi_0(x)) + \sum_{k=1}^{\infty} \xi_k \varphi_k(x) \tag{9.29}$$

with $\varphi_k = \tilde{\varphi}_k \sqrt{\lambda_k}$ as eigenpairs of the correlation function C

$$C \tilde{\varphi}_k = \lambda_k \tilde{\varphi}_k. \tag{9.30}$$

Due to the representation we can write

$$f(x_i, \xi) = \sum_{k \in \mathbb{N}} \xi_k \psi_k(x_i) = \Phi^T \xi \tag{9.31}$$

and we can calculate the unknown density p, given a linear measurement operator

$$\mathcal{O}_i f(\cdot,\xi) = f(x_i,\xi), \quad \text{for } i = 1,\dots,p, \tag{9.32}$$

by Bayes formula

$$\begin{aligned}
p(\xi) = \frac{1}{Z}e^{-\Phi(\xi)} &= \frac{1}{Z}e^{-\frac{1}{2}\sigma\sum_{i=1}^{p}|y_i-\mathcal{O}_i f(\cdot,\xi)|^2}e^{-\lambda\langle\xi,\Gamma\xi\rangle} && (9.33)\\
&= \frac{1}{Z}e^{-\frac{1}{2}\sigma\left(\langle y-\Phi^T\xi, y-\Phi^T\xi\rangle+\lambda\langle\xi,\Gamma\xi\rangle\right)} && (9.34)\\
&= \frac{1}{Z}e^{-\sigma\left(\frac{1}{2}\langle y,y\rangle-\langle\Phi y,\xi\rangle+\frac{1}{2}\langle(\Phi\Phi^T+2\lambda I)\xi,\xi\rangle\right)} && (9.35)\\
&= \frac{1}{Z}\left(e^{-\frac{\sigma}{2}\|y\|^2}e^{-\sigma(-\langle\Phi y,\xi\rangle+\langle Q\xi,\xi\rangle)}\right) && (9.36)\\
& && (9.37)
\end{aligned}$$

where we can apply quadratic expansion in the sense

$$\langle Q(\xi-\xi_0),\xi-\xi_0\rangle = \langle Q\xi_0,\xi_0\rangle - 2\langle Q\xi_0,\xi\rangle + \langle Q\xi,\xi\rangle \tag{9.38}$$

which leads, by coefficient comparison to the representation

$$Q := \frac{1}{2}\left(\Phi\Phi^T + 2\lambda I\right). \tag{9.39}$$

$$Q\xi_0 = \Phi y \tag{9.40}$$

and

$$\frac{1}{2}\left(\Phi\Phi^T + 2\lambda I\right)\xi_0 = \Phi y, \tag{9.41}$$

which is the equivalent formulation of yesterday. Moreover,

$$\xi_0 = 2\left(\Phi\Phi^T + 2\lambda I\right)^{-1}\Phi y. \tag{9.42}$$

can be seen as

$$\xi_0 = \operatorname{argmin} J(\xi) = \operatorname{argmin} \frac{1}{2}\sum_{i=1}^{p}|y_i - f(x_i,\xi)|^2 + \frac{\lambda}{2}\langle\xi,\xi\rangle \tag{9.43}$$

Originally, we were interested in calculating the expectation of the conditional random variable $(y|x)$. Substituting

$$\eta_k = \xi_k - \xi_{0,k} \tag{9.44}$$

$$\xi_k = \eta_k + \xi_{0,k} \tag{9.45}$$

yields

$$\begin{aligned}
\int f(x,\xi)p(\xi)d\xi &= \int (\sum_{k\in\mathbb{N}}\xi_k\varphi_k(x))\frac{1}{Z}e^{-\langle Q(\xi-\xi_0),\xi-\xi_0\rangle}d\xi && (9.46)\\
&= \sum_{k\in\mathbb{N}}\varphi_k(x)\frac{1}{Z}\int \xi_k e^{-\langle Q(\xi-\xi_0),(\xi-\xi_)\rangle}d\xi && (9.47)\\
&= \sum_{k\in\mathbb{N}}\varphi_k(x)\xi_{0,k}\underbrace{\frac{1}{Z}\int e^{\langle Q\eta,\eta\rangle}d\eta}_{=1} + \underbrace{\frac{1}{Z}\int \eta_k e^{\langle Q\eta,\eta\rangle}d\xi)}_{=0} && (9.48)\\
&= \sum_{k\in\mathbb{N}}\varphi_k(x)\xi_{0,k} = f(x,\xi_0). && (9.49)
\end{aligned}$$

As a summary

Remark 9.12.

1. This technique yields the same results as the Kernel Ridge regression but
2. The kernel and feature space $(\varphi_k)_k$ comes from the Karhunen-Loeve expansion and has a statistical interpretation.
3. The prior distribution corresponds to the regularisation term in the modified minimization problem.

Appendix

A Stochastic Processes

A.1 Random Walk and Brownian Motion

1.1 **Example A.1** (Random Walk)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a sequence $(X_n)_{n\in\mathbb{N}_0}$ of random variables (r.v.) on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{Z}$ defined by*

$$\begin{cases} X_1 := 0, \\ X_{n+1} := X_n + \xi_n, \qquad n \in \mathbb{N} \end{cases}$$

where $(\xi_n)_{n\in\mathbb{N}}$ is in turn a sequence of i.i.d. r.v. with $\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = \frac{1}{2}$ for $n \in \mathbb{N}$.

Then, for arbitrary $n \in \mathbb{N}$, we have $\mathbb{E}[X_n] = 0$, since also $\mathbb{E}[\xi_j] = 0$ for all $j \in \mathbb{N}$. Moreover, $\mathbb{E}[\xi_i \xi_j] = \delta_{i,j}$ for arbitrary $i, j \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{Cov}[X_n, X_m] &:= \mathbb{E}[(X_n - \mathbb{E}[X_n])(X_m - \mathbb{E}[X_m])] \\ &= \mathbb{E}[X_n X_m] = \mathbb{E}\left[\left(\sum_{j=1}^{n-1} \xi_j\right)\left(\sum_{k=1}^{m-1} \xi_k\right)\right] \\ &= \min\{n-1, m-1\}. \end{aligned}$$

Now we define $X(t)$ to be the piecewise linear interpolation of $(X_n)_{n\in\mathbb{N}}$. This gives a set of random variables indexed by $t \in \mathbb{R}_0^+$. We will call $X(t)$ a stochastic process*.*

1.2 **Definition A.2** (Stochastic process)**.** Given a set $\mathcal{T} \subset \mathbb{R}$, a measurable space $(H, \mathcal{H})$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A family $X = \{X(t) : t \in \mathcal{T}\}$ is called *H-valued stochastic process* if for every $t \in \mathcal{T}$ the function $X(t)$ is $\mathcal{F}$-$\mathcal{H}$-measurable.

1.3 **Definition A.3** (Sample Path)**.** For a given $\omega \in \Omega$, the mapping defined by

$$X(\cdot, \omega) \colon T \to H,\ t \mapsto X(t, \omega) \tag{A.1}$$

is called *sample path* of X.

1.4 *Remark* A.4. Diffusion scaling of X as in Example A.1 leads to a Brownian motion.

1.5 **Example A.5** (Distribution of a random walk)**.** *With the notation from example A.1 let $p_{n,j} = \mathbb{P}(X_n = j)$, $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$. Then, we have for arbitrary $n \in \mathbb{N}$ and $j \in \mathbb{Z}$*

$$p_{n+1,j} = \frac{1}{2}(p_{n,j-1} + p_{n,j+1})$$

as well as $p_{0,j} = \delta_{0,j}$.

Note that $p_{2n,2k+1} = 0$ for arbitrary $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.

1.6 **Lemma A.6.** *With the notation of examples A.1 and A.5 we have*

(i) $\frac{1}{n} X_n \to 0$ $\mathbb{P}$*-a.s.*

(ii) $\frac{1}{\sqrt{n}} X_n \xrightarrow{Law} \mathcal{N}(0,1)$.

Proof. We have $X_n = \sum_{j=1}^{n-1} \xi_j$ with $\mathbb{E}[\xi_j] = 0$ and $\mathbb{V}[\xi_j] = 1$ for all $j \in \mathbb{N}$. Since $(\xi_n)_{n\in\mathbb{N}}$ are i.i.d. Bernoulli RVs (centered around zero), the two statements are exactly the law of large numbers and the central limit theorem. □

The random walk is approximately Gaussian for n large. We derive a sequence of stochastic procceses $Y_N = \{Y_N(t) : t \in \mathbb{R}^n\}$ for $N \in \mathbb{N}$ where $Y_N(t)$ is approximately Gaussian for large N.

Let again, X be the linear interpolation of a random walk $(X_n)_{n\in\mathbb{N}}$. Now, we define

$$Y_N(t) := \frac{1}{\sqrt{N}} X(tN) = \frac{1}{\sqrt{N}} X_{\lfloor tN \rfloor} + \frac{tN - \lfloor tN \rfloor}{\sqrt{N}} \xi_{\lfloor tN \rfloor}.$$

Then, $Y_N = \{Y_N(t) : t \in \mathbb{R}_0^+\}$ is a real-valued stochastic process for each $N \in \mathbb{N}$ with continuous sample paths.

1.7 **Lemma A.7.** *With Y_N as above, we have for arbitrary $t \in \mathbb{R}_0^+$, $Y_N(t) \xrightarrow{Law} \mathcal{N}(0,t)$ as $N \to \infty$.*

Proof. The second term in the definition of $Y_N(t)$ converges to zero in probability, i.e. for every $\varepsilon > 0$

$$\mathbb{P}\left(\left| \frac{tN - \lfloor tN \rfloor}{\sqrt{N}} \xi_{\lfloor tN \rfloor} \right| > \varepsilon \right) \xrightarrow{N\to\infty} 0.$$

The first term in the definition converges to the desired distribution by Lemma A.6. Since convergence in probability implies convergence in distribution, we are already done. □

1.8 *Remark* A.8. For every $t \in \mathbb{R}_0^+$ we have $Y_N(t) \in L^2(\Omega)$ for each $N \in \mathbb{N}$. However, the sequence $(Y_N(t))_{N\in\mathbb{N}}$ is not Cauchy, i.e. it has no limit in $L^2(\Omega)$.

1.9 **Definition A.9** (Finite-Dimensional Distributions)**.** Let $X = \{X(t) : t \in \mathcal{T}\}$ be a real-valued stochastic process. For $t_1, \ldots, t_M \in \mathcal{T}$ we define

$$\mathbf{X} := (X(t_1), \ldots, X(t_M))^T.$$

Then $\mathbf{X}$ is a $\mathbb{R}^M$-valued r.v. and the probability distribution $\mathbb{P}_{\mathbf{X}}$ on $(\mathbb{R}^M, \mathcal{B}(\mathbb{R}^M))$ is known as the *finite-dimensional distribution* of X at $t_1, \ldots, t_M$.

1.10 **Lemma A.10.** *For the diffusion scaled random walk Y_N as above and $\mathbf{Y}_N := (Y_N(t_1), \ldots, Y_N(t_M))^T$ a finite-dimensional distribution we have $\mathbf{Y}_N \xrightarrow{Law} \mathcal{N}(0, C)$ as $N \to \infty$, where $C \in \mathbb{R}^{M\times M}$ with entries $c_{i,j} = \min\{t_i, t_j\}$.*

1.11 **Definition A.11** (Second Order Process)**.** A stochastic process $X = \{X(t) : t \in \mathcal{T}\}$ is said to be of *second order*, if $X(t) \in L^2(\Omega)$ for all $t \in \mathcal{T}$.

If X is of second order, we define the *mean function*

$$\mu : t \mapsto \mathbb{E}[X(t)] \tag{A.2}$$

and the *covariance function*

$$C : (s,t) \mapsto \operatorname{Cov}(X(s), X(t)) \tag{A.3}$$

of X.

1.12 **Definition A.12** (Real-valued Gaussian Process)**.** A real-valued stochastic process X is called *Gaussian process*, if all its finite-dimensional distributions are multivariate Gaussian distributions.

1.13 **Definition A.13** (Brownsche Bewegung)**.** The processes $W = \{W(t) : t \in \mathbb{R}_0^+\}$ is called *Brownian motion*, if it is a real-valued Gaussian process with continuous sample paths, mean function $\mu(t) = 0$ and covariance function $C(s,t) = \min\{s,t\}$.

1.14 *Remark* A.14.

- The Browniam motion W can be understood as limiting process of the rescaled linear interpolants Y_N.
- As with a random walk, the increments $W(t)-W(s)$ over disjoint intervals are independent. In particular, for $p \le r \le s \le t$ we find

$$\mathrm{Cov}[W(r) - W(p), W(t) - W(s)] = 0. \tag{A.4}$$

  Since W is a Gaussian process, for which correlation zero already shows independence, we showed that increments of the Brownian motion are independent.

  We can further calculate $\mathbb{V}[W(t) - W(s)] = |t-s|$, which implies that $W(t) - W(s) \sim \mathcal{N}(0, |t-s|)$.

- An alternative definition of the Brownian motion is: $W = \{W(t) : t \in \mathbb{R}_0^+\}$ is a real valued stochastic procces with
  - (i) $W(0) = 0$ a.s.
  - (ii) $W(t) - W(s) \sim \mathcal{N}(0, |t-s|)$ for all $t, s \in \mathbb{R}_0^+$,
  - (iii) for $p, r, s, t \in \mathbb{R}$ with $p \le r \le s \le t$ we have $W(r) - W(p)$ and $W(t) - W(s)$ are independent
  - (iv) The paths of W are almost surely continuous.

1.15 *Remark* A.15 (Relation of the random walk and the heat equation)**.** Rescale the random walk (X_n) to $(Y_n) = \left(\frac{1}{\sqrt{N}} X_n\right)$ for some $N \in \mathbb{N}$. On the lattice $(t_n, x_n) = (n\Delta t_j, \Delta x)$, $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$ with $\Delta t = \Delta x^2 = \frac{1}{N}$. As above, we have $q_{n,j} = \mathbb{P}(Y_n = j)$ satisfies

$$q_{n+1,j} := \frac{1}{2}(q_{n,j-1} + q_{n,j+1})$$

and thus

$$\frac{q_{n+1,j} - q_{n,j}}{\Delta t} = \frac{1}{2} \frac{q_{n,j-1} - 2q_{n,j} + q_{n,j+1}}{\Delta x^2}.$$

This is equivalent to the finite difference approximation of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}, \qquad x \in \overline{\mathbb{R}}, t \in \mathbb{R}_0^+ \tag{5}$$ eqn:diffusionProblem

for initial conditions $u(0,x) = \varphi(x)$, $\varphi : \mathbb{R} \to \mathbb{R}$.

We define

$$U_{n,j} := \mathbb{E}[\varphi(x_j + Y_n)] = \sum_{k \in \mathbb{Z}} \varphi(x_j + x_k) q_{n,k}$$

By the above definition of $q_{n,j}$, we have

$$U_{n+1,j} = \sum_{k\in\mathbb{Z}} \varphi(x_j + x_k) q_{n+1,k} = \sum_{k\in\mathbb{Z}} \varphi_{x_j+x_k} \cdot \frac{1}{2}(q_{n,k-1} + q_{n,k+1})$$
$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} (\varphi(x_j + x_{k+1}) + \varphi(x_j + x_{k-1}))q_{n,k} = \frac{1}{2}(U_{n,j+1} + U_{n,j-1})$$

This shows

$$\frac{U_{n+1,j} - U_{n,j}}{\Delta t} = \frac{1}{2}\frac{U_{n,j-1} - 2U_{n,j} + U_{n,j+1}}{\Delta x^2} \qquad (6)$$ eqn:discreteDiffusion

with starting value $U_{0,j} = \varphi(x_j)$.

The finite differences (FD) discretization of (5) with initial condition $u(0, x) = \varphi(x)$ on the other hand is given by

$$\frac{u_{n+1,j} - u_{n,j}}{\Delta t} = \frac{1}{2}\frac{u_{n,j-1} - 2u_{n,j} + u_{n,j+1}}{\Delta x^2}, \qquad u_{0,j} = \varphi(x_j) \qquad (7)$$ eqn:diffusionFDdiscretiz

Since in (6) and (7) update rules and initial conditions are the same, the FD approximation $u_{n,j}$ and the $U_{n,j}$ of the random walk are equivalent.

1.16 **Theorem A.16.** *For continuous bounded initial data φ, the FD approximation $u_{n,j}$ defined by (7) converges to the solution of (5) as $\Delta t = \Delta x^2 \to 0$.*

Proof. Recall that $Y_n := Y_n(t_n)$, then $Y_n \xrightarrow{\text{Law}} \mathcal{N}(0, t_n)$ by Lemma A.10. Hence, $x_j + Y_n \xrightarrow{\text{Law}} \mathcal{N}(x_j, t_n)$ as $N \to \infty$ with (t_n, x_j) fixed.

Equivalently,

$$U_{n,j} = \mathbb{E}[\varphi(x_j + Y_n)] \to \int_{\mathbb{R}} \varphi(x) \cdot \frac{1}{\sqrt{2\pi t_n}} e^{-\frac{(x-x_j)^2}{2t_n}} \,\mathrm{d}x.$$

It is well known, that the exact solution of (5) is given by

$$u(t, x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2t}} \varphi(y) \,\mathrm{d}y.$$

Since, $U_{n,j} = u_{n,j}$, this yields $u_{n,j} \to u(t_n, x_j)$ for $N \to \infty$ and fixed (t_n, x_j). □

1.17 *Remark* A.17.

- The distribution of a real-valued multivariate Gaussian r.v. is uniquely determined by its mean vector μ and covariance matrix C. Here C must be symmetric and non-negative definite.
- Similarly for a Gaussian process $X = \{X(t) : t \in \mathcal{T}\}$, the covariance function $C : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ must be symmetric and non-negative definite, i.e. for arbitrary $N \in \mathbb{N}$, $t_1, \ldots, t_N \in \mathcal{T}$ and $a_1, \ldots, a_N \in \mathbb{R}$ we always have

$$\sum_{j,k=1}^{N} a_j C(t_j, t_k) a_k \geq 0.$$

- Let $\mathbb{R}^{\mathcal{T}}$ be the set of all functions $f : \mathcal{T} \to \mathbb{R}$ and $\mathcal{B}(\mathbb{R}^{\mathcal{T}})$ the smallest σ-field containing all open sets of $\mathbb{R}^{\mathcal{T}}$. For $t_1, \ldots, t_N \in \mathcal{T}$ and $F \in \mathcal{B}(\mathbb{R}^{\mathcal{T}})$, we define

$$B := \{f \in \mathbb{R}^{\mathcal{T}} : (f(t_1), \ldots, f(t_N))^T \in F\}.$$

Note that Definition A.3 implies that the sample paths $X(\cdot, \omega)$ of a real-valued stochastic process $X = \{X(t) : t \in \mathcal{T}\}$ belong to $\mathbb{R}^{\mathcal{T}}$. Moreover, the sample paths define a $\mathbb{R}^{\mathcal{T}}$-valued r.v.

1.19 **Definition A.18** (Independet processes, sample paths). (i) Two real-valued stochastic processes $X = \{X(t) : t \in \mathcal{T}\}$ and $Y = \{Y(t) : t \in \mathcal{T}\}$ are called *independent* if the associated $(\mathbb{R}^{\mathcal{T}}, \mathcal{B}(\mathbb{R}^{\mathcal{T}}))$-valued r.v. are independent, i.e.

$$\mathbf{X} = (X(t_1), \dots, X(t_N))^T \quad \text{and} \quad \mathbf{Y} = (Y(s_1), \dots, Y(s_N))^T$$

are independent $\mathbb{R}^N$-valued r.v. for arbitrary $t_1, \dots, t_N, s_1, \dots, s_N \in \mathcal{T}$.

(ii) The functions $f : \mathcal{T} \to \mathbb{R}$, $i \in \{1, 2\}$, are inpendent sample paths of a real-valued processes $X = \{X(t) : t \in \mathcal{T}\}$ if $f_i(t) = X_i(t, \omega)$ for some $\omega \in \Omega$, where X_i are i.i.d. processes with the same distribution as X.

1.20 **Theorem A.19** (Daniel-Kolmogorov). *Let $\mathcal{T} \subset \mathbb{R}$. The following statements are equivalent.*

(i) There exists a real-valued second order stochastic processes with mean function μ and Covariance function $C(s, t)$.

(ii) $\mu : \mathcal{T} \to \mathbb{R}$, $C : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ with C symmetric and non-negative definite.

1.21 **Corollary A.20.** *The probability distribution $\mathbb{P}_X$ on $(\mathbb{R}^{\mathcal{T}}, \mathcal{B}(\mathbb{R}^{\mathcal{T}}))$ of a real-valued Gaussian processes is uniquely determined by its mean function μ and covariance function C.*

1.22 **Example A.21** (Cosine Covariance). *Let $C(s, t) = \cos(s - t)$ and $\mu \equiv 0$. Obviously C is symmetric and it is easy to show that for $t_1, \dots, t_N \in \mathcal{T} = \mathbb{R}$, $a_1, \dots, a_n \in \mathbb{R}$*

$$\sum_{j,k=1}^{N} a_j \cos(t_j - t_k) a_k = \left| \sum_{j=1}^{N} a_j \cos(t_j) \right|^2 + \left| \sum_{j=1}^{N} a_j \sin(t_j) \right|^2 \geq 0.$$

In fact, $X(t) = \xi_i \cos(t) + \xi_2 \sin(t)$ with $\xi_1, \xi_2 \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$ is the Gaussian processes defined by μ and C above.

For the existance of a *Brownian Motion* we need the following

1.23 **Lemma A.22.** *$C : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}, (s, t) \mapsto \min\{s, t\}$ is symmetric and non-negative definite.*

A.2 Conditional Expectation for Square Integrable H-valued r.v.

1.24 **Definition A.23** ($L^2(\Omega, \mathcal{F}, \mathbb{P})$). For a Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$, the space of $\mathcal{F}$-measurable r.v. from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(H, \mathcal{B}(H))$ with $\mathbb{E}[\|X\|^2] < \infty$ is denoted by $L^2(\Omega, \mathcal{F}, \mathbb{P}; H) =: L^2(\Omega; H)$ and forms a Hilbert space with the inner product

$$\langle X, Y \rangle_{L^2(\Omega;H)} = \mathbb{E}\left[\langle X, Y \rangle_H\right]$$

for $X, Y \in L^2(\Omega; H)$.

1.25 **Definition A.24** (Conditional Expectation Given a σ-field)**.** Let $X \in L^2(\Omega; H)$ and $\mathcal{A}$ a sub-σ-field of $\mathcal{F}$. Then, the **conditional expectation** of X given $\mathcal{A}$, denoted by $\mathbb{E}[X|\mathcal{A}]$, is defined as orthogonal projection of X onto the space $L^2(\Omega, \mathcal{A}; H) \subset L^2(\Omega, \mathcal{F}; H)$.

Consider a Hilbert space Ψ and a σ-field $\mathcal{G}$. The probability distribution $\mathbb{P}_X$ of a Ψ-valued r.v. X is

$$\mathbb{P}_X(G) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in G\}) = \mathbb{E}[\mathbb{1}_G(X)], \qquad \text{for } G \in \mathcal{G}.$$

Now we can use the definition of conditional expectations to define also conditional probability distributions.

1.26 **Definition A.25** (Conditional Probability)**.** For a Ψ-valued r.v. X on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{A}$ a sub-σ-field of $\mathcal{F}$ we define

$$\mathbb{P}_X[G|\mathcal{A}] := \mathbb{E}[\mathbb{1}_G(X)|\mathcal{A}].$$

Then $\mathbb{P}_X[G|\mathcal{A}]$ is a $[0,1]$-valued r.v. and $G \mapsto \mathbb{P}_X[G|\mathcal{A}]$ is a measure-valued r.v.

For H-valued r.v. Y and $y \in H$, define

$$\mathbb{P}_X[G|Y = y] := \mathbb{P}[X \in G|Y = y] = \mathbb{E}[\mathbb{1}_G(X)|Y = y].$$[1]

The map $G \mapsto \mathbb{P}_X[G|Y = y]$ is a measure on $(\Psi, \mathcal{G})$. Then $\mathbb{P}_X[G|Y = y]$ is called conditional probability that $X \in G$ given $Y = y$.

1.27 **Example A.27** (Brownian Bridge)**.** *Let $\mathcal{T} = [0, T]$ for some fixed $T \in \mathbb{R}^+$. We want to define a new stochastic process, the Brownian bridge (*Bb*), which behaves like a standard Brownian motion restricted to $\mathcal{T}$, except for the fact that we impose that its final point shall be equal to a beforehand fixed value $b \in \mathbb{R}$.*

We realise this by conditioning W to $W(0) = 0$ (as it is already defined) and $W(T) = b$. Hence the finite-dimensional distribution of a Bb *B at $t_1, \ldots, t_N \in \mathcal{T}$ are given by*

$$\mathbb{P}\left[\begin{pmatrix} B(t_1) \\ \vdots \\ B(t_N) \end{pmatrix} \in F\right] = \mathbb{E}\left[\mathbb{1}_F\begin{pmatrix} W(t_1) \\ \vdots \\ W(t_N) \end{pmatrix}\middle| W(T) = b\right],$$

i.e. for measurable $\varphi : \mathbb{R}^N \to \mathbb{R}$ we have

$$\mathbb{E}[\varphi(B(t_1), \ldots, B(t_N))] = \mathbb{E}[\varphi(W(t_1), \ldots, W(t_n))|W(T) = b].$$

From a more indepth study of conditional expression and the resulting disintegration theorem, one sees that a thusly defined process is again Gaussian and we can calculate the mean and covariance functions of the Bb *as $\mu(t) = \mathbb{E}[W(t)|W(T) = b]$ and $C(s,t) = \mathbb{E}[(W(s) - \mu(s))(W(t) - \mu(t)|W(T) = b]$.*

1.28 **Lemma A.28** (Standard Brownian Bridge)**.** *The stochastic process $B = \{B(t) : t \in [0,1]\}$ with $B(0) = 0 = B(1)$ is a Gaussian process with mean function $\mu \equiv 0$ and $C : (s,t) \mapsto \min\{s,t\} - s \cdot t$.*

[1] More precisely, the definition is based on the following

Lemma A.26. *A r.v. Y is $\sigma(X)$-measurable if and only if there exists a measurable function f such that $Y = f(X)$. This function is $\mathbb{P}_X$-almost surely unique.*

Hence there exists a unique function f such that $\mathbb{E}[Y|X] := \mathbb{E}[Y|\sigma(X)] = f(X)$ $\mathbb{P}$-a.s. Now we simply define symbolically $\mathbb{E}[Y|X = x] = f(x)$.

A.3 White and Coloured Noise

1.29 *Remark* A.29 (White Noise). Let $\mathcal{T} = [0, 1]$ and $(\varphi_n)_{n\in\mathbb{N}}$ be a orthonormal basis of $L^2(\mathcal{T})$. For instance

$$\varphi_j(t) = \sqrt{2}\sin(j\pi t). \qquad (*)$$ eqn:basisFkt

Consider the stochastic process ζ (the **wite noise**) given by

$$\zeta(t) = \sum_{j=1}^{\infty} \xi_j \varphi_j(t) \qquad (**)$$ eqn:summingProcess

with $(\xi_j)_{j\in\mathbb{N}}$ a sequence of i.i.d. $\mathcal{N}(0, 1)$-r.v. By definition, ζ has mean function $\mu \equiv 0$ and the covariace

$$C(s, t) = \operatorname{Cov}[\zeta(s), \zeta(t)] = \sum_{j,k=1}^{\infty} \operatorname{Cov}[\zeta_j, \zeta_k]\varphi_j(s)\varphi_k(t) = \sum_{j,k=1}^{\infty} \varphi_j(s)\varphi_k(t).$$

For the basis $(*)$, we get $C(s, t) = \delta(s - t)$. Hence, for $s \neq t$, the covariance vanishes, i.e. the process is self-uncorrelated and for $s = t$ we have

$$C(t, t) = \mathbb{V}[\zeta(t)] = \mathbb{E}[\zeta(t)^2] = \delta(0) = \infty.$$

In particular, the process ζ does not converge and is not in $L^2(\mathcal{T})$.

1.30 *Remark* A.30 (Coloured Noise). With the notation from Remark (A.29) define a processes X on $\mathcal{T}$ via

$$X(t) = \sum_{j=1}^{\infty} \sqrt{\nu_j}\xi_j\varphi_i(t).$$

If ν_j vary with $j \in \mathbb{N}$, then $X(t)$ is called **coulered noise** and the random variables $X(t)$ and $X(s)$ are correlated.

A.4 Karhunen-Loève expansion

1.31 *Remark* A.31 (Matrix Spectral Decomposition, Discrete KLE). Let $X = \{X(t) : t \in \mathcal{T}\}$ a real-valued Gaussian processes with mean function μ and covariance function C. For $t_1, \dots, t_N \in \mathcal{T}$ define

$$\mathbf{X} = (X(t_1), \dots, X(t_N))^T \sim \mathcal{N}(\mu, C_N) \qquad \text{(A.5)}$$

with $\mu = (\mu(t_1), \dots, \mu(t_N))^T$ and $C_N \in \mathbb{R}^{N\times N}$ with entries $c_{i,j} = C(t_i, t_j)$. Samples of X can be generated with $C_N = V^T V$ (Cholesky decomposition of C_N) and

$$X = \mu + V^T\xi \qquad (8)$$ eqn:8

with $\xi = (\xi_1, \dots, \xi_N)^T \sim \mathcal{N}(0, I_N)$ which can be generated as N i.i.d. normal r.v. (e.g. via Box-Miller).

1.32 **Theorem A.32** (Spectral Decomposition)**.** *Every symmetric $A \in \mathbb{R}^{N\times N}$ can be decomposed as $A = U\Sigma U^T$ where U is orthonormal with columns u_j, which are eigenvectors of A and $\Sigma = \operatorname{diag}(\nu_1, \dots, \nu_N)$ the associated eigenvalues of A.*

1.33 *Remark* A.33.
- The advantage of the spectral decomposition is, that one can use the singular value decomposition, which is more robust than the Cholesky decomposition.

- Since the covariance matrix C_N in Remark A.31 is square, real-valued and symmetric we have $C_N = U\Sigma U^T$ and since C_N is non-negative definite, we can order the eigenvalues of C_N as $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_N \geq 0$. Let now

$$X = \mu + \sum_{j=1}^{N} \sqrt{\nu_j} u_j \xi_j,$$

with $\xi_j \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

- Efficiency gained by truncation of spectral decomposition: In the decomposition $C_N = \sum_{j=1}^{N} \nu_j u_j u_j^T$ the first terms (those with biggest eigenvalues) contribute most. For $n < N$ define $\Sigma_n := \operatorname{diag}(\nu_1, \ldots, \nu_n)$ and $U_n = (u_1, \ldots, u_n) \in \mathbb{R}^{n \times N}$. Then we define the truncated spectral decomposition

$$C_{N,n} = U_n \Sigma_n U_n^T = \sum_{j=1}^{n} \nu_j u_j u_j^T.$$

The approixmation error yields

$$\|C_N - C_{N,n}\|_{L(\mathbb{R}^N)} = \sup_{x \neq 0} \frac{\|(C_N - C_{N,n})x\|_2}{\|x\|_2} = \left\| \sum_{j=n+1}^{N} \nu_j u_j u_j^T \right\|_{L(\mathbb{R}^N)} = \nu_{n+1}.$$

- The truncation of (8) gives

$$\hat{X} = \mu + U_n \Sigma_n^{\frac{1}{2}} \xi = \mu + \sum_{j=1}^{n} \sqrt{\nu_j} u_j \xi_j$$

with $\xi = (\xi_1, \ldots, \xi_n) \sim \mathcal{N}(0, I_n)$, i.e. $\hat{X} \sim \mathcal{N}(\mu, C_{N,n})$. If X is defined by (8), then we have

$$\mathbb{E}\left[\left\|X - \hat{X}\right\|_2^2\right] = \mathbb{E}\left[\sum_{j=n+1}^{N} \sum_{k=n+1}^{N} \sqrt{\nu_j \nu_k} u_j^T u_k \xi_j \xi_k\right] = \sum_{j=n+1}^{N} \nu_j.$$

The error in approximation averages with respect to test functions $\varphi \in L^2(\mathbb{R}^N)$ satisfies

$$\left|\mathbb{E}[\varphi(X)] - \mathbb{E}\left[\varphi\left(\hat{X}\right)\right]\right| \leq k \|\varphi\|_{L^2} \|C_N - C_{N,n}\|_F^2 \leq k \|\varphi\|_{L^2} \sum_{j=n+1}^{N} \nu_j^2.$$

- The decay of the eigenvalues of C_N determines the approximability by few terms.

- Trunctation is a method for model order reduction: Approximating a N-dimensional Gaussian r.v. X is approximated by an n ($< N$) uncorrelated $\mathcal{N}(0,1)$-r.v.

1.34 *Remark* A.34 (Karhunen-Loève Expansion). A generalization of A.31 to A.33 to stachastic processes $X = \{X(t) : t \in \mathcal{T}\}$ is called KLE. Let $\mu(t) = \mathbb{E}[X(t)]$ and consider $Y(t) = X(t) - \mu(t)$ (the centering of X).

Our aim is to sample paths in an orthonormal basis $(\varphi_j)_{j\in\mathbb{N}}$ of $L^2(\mathcal{T})$, i.e.

$$X(t,\omega) - \mu(t) = \sum_{j=1}^{\infty} \gamma_j(\omega)\varphi_j(t)$$

with r.v. $\gamma_j(\omega) := \langle X(t,\omega) - \mu(t), \varphi_j(t)\rangle_{L^2(\mathcal{T})}$. In the KLE we chose $(\varphi_j)_{j\in\mathbb{N}}$ as the eigenfunctions of the integral operator $\mathcal{C}$, which is defined by

$$(\mathcal{C}f)(t) := \int_{\mathcal{T}} C(s,t)f(s)\,\mathrm{d}t, \qquad \text{for } f \in L^2(\mathcal{T}).$$

1.35 **Lemma A.35.** *Let $X \in L^2(\Omega; L^2(\mathcal{T}))$. Then, $\mu \in L^2(\mathcal{T})$ and the sample paths $X(t,\omega) \in L^2(\mathcal{T})$ a.s. .*

Proof. By assumption $\|X\|_{L^2(\Omega;L^2(\mathcal{T}))} = \mathbb{E}[\|X\|^2_{L^2(\mathcal{T})}] < \infty$ and $\|X(t,\omega)\|_{L^2(\mathcal{T})} < \infty$ a.s. $w \in \Omega$ and sample paths $X(\cdot,\omega) \in L^2(\mathcal{T})$ a.s. Jensen's inequality gives $\mu(t)^2 = \mathbb{E}[X(t)]^2 \leq \mathbb{E}[X(t)^2]$ and hence

$$\|\mu\|^2_{L^2(\mathcal{T})} = \int_{\mathcal{T}} \mu(t)\,\mathrm{d}t \leq \int_{\mathcal{T}} \mathbb{E}[X(t)^2]\,\mathrm{d}t = \|X\|^2_{L^2(\Omega;L^2(\mathcal{T}))} < \infty.$$

□

1.36 **Theorem A.36** (L^2-Convergence of the KLE)**.** *Consider $X = \{X(t) : t \in \mathcal{T}\}$ and suppose $X \in L^2(\Omega; L^2(\mathcal{T}))$. Then*

$$X(t,\omega) = \mu(t) + \sum_{j=1}^{\infty} \sqrt{\nu_j}\varphi_j(t)\xi_j(\omega) \tag{10}$$ eqn:10

converges in $L^2(\Omega; L^2(\mathcal{T}))$, with

$$\xi_j(\omega) = \frac{1}{\sqrt{\nu_j}} \langle X(t,\omega) - \mu(t), \varphi_j(t)\rangle_{L^2(\mathcal{T})},$$

and $\{\nu_j, \varphi_j\}_{j\in\mathbb{N}}$ denotes eigenpairs of the integraloperator $\mathcal{C}$ with kernel function C, the covariance function of X, with $\nu_i \geq \nu_{n+1} \geq 0$ for all $n \in \mathbb{N}$. The ξ_n have mean zero, variance one and are pairwise uncorrelated. If X is a Gaussian process, we have $\xi_j \sim \mathcal{N}(0,1)$.

1.37 **Theorem A.37** (Uniform Convergence of the KLE)**.** *Consider a real valued stochastic process $X \in L^2(\Omega; L^2(\mathcal{T}))$ and let $X_J(t,\omega)$ be defined as*

$$X_J(t,\omega) := \mu(t) + \sum_{j=1}^{J} \sqrt{\nu_j}\varphi_j(t)\xi_j(\omega).$$

If $\mathcal{T} \subset \mathbb{R}$ is a compact set and the covariance function C of X belons to $C(\mathcal{T}\times\mathcal{T})$, then $\varphi_j \in C(\mathcal{T})$ and the expansion of C converges uniformly, i.e.

$$\sup_{s,t\in\mathcal{T}} |C(s,t) - C_J(s,t)| \leq \sup_{t\in\mathcal{T}} \sum_{j=J+1}^{\infty} \nu_j\varphi_j(t)^2 \to 0, \qquad \text{for } J \to \infty,$$

where $C_J = \sum_{j=1}^{J} \nu_j\varphi_j(t)\varphi_j(s)$ is the covariance function of X_J and

$$\mathbb{E}[|X(t) - X_J(t)|^2] \to 0, \qquad \text{for } J \to \infty.$$

A.5 Regularity of Stochastic Processes

1.38 **Definition A.38** (Mean-Square Continuity)**.** A stochastic process $X = \{X(t) : t \in \mathcal{T}\}$ is **mean-square continuous**, if, for all $t \in \mathcal{T}$,

$$\mathbb{E}\left[(X(t+h) - X(t))^2\right] \to 0 \qquad \text{as } h \to 0.$$

1.39 **Theorem A.39.** *Let $X = \{X(t) : t \in \mathcal{T}\}$ be a centered processes. The covariance function C of X is continuous at $(t,t) \in \mathcal{T} \times \mathcal{T}$ if and only if X is mean-square continuous in t. In particular, if $C \in C(\mathcal{T} \times \mathcal{T})$, then X is mean-square continuous everywhere.*

1.40 **Definition A.40** (Mean-Square Derivative)**.** A stochastic process $X = \{X(t) : t \in \mathcal{T}\}$ is **mean-square differentiable** with mean-square derivative $\frac{\mathrm{d}X(t)}{\mathrm{d}t}$, for all $t \in \mathcal{T}$, if

$$\mathbb{E}\left[\left|\frac{X(t+h) - X(t)}{h} - \frac{\mathrm{d}X(t)}{\mathrm{d}t}\right|^2\right]^{\frac{1}{2}} \to 0, \qquad \text{as } h \to 0.$$

1.41 **Theorem A.41.** *Given a centered stochastic-process X with covariance function $C \in C^2(\mathcal{T}, \mathcal{T})$. Then, X is mean-square differentiable in any $t \in \mathcal{T}$ and the derivative $\frac{\mathrm{d}X(t)}{\mathrm{d}t}$ has the covariance function $\frac{\partial^2 C(s,t)}{\partial s \partial t}$.*

1.42 *Remark* A.42. Analogously, if it exist, the r-th mean-square derivative has covariance function $\frac{\partial^{2r} C(s,t)}{\partial s^r \partial t^r}$.

1.43 **Definition A.43** (Stationary Process)**.** A stochastic processes X is called **(weakly) stationary**, if it has constant mean function $\mu(t) \equiv \mu$ and its covariance function $C(s,t)$ depends only on the difference $s - t$, i.e. $C(s,t) = C(s-t, 0) = c(s-t)$ for some so called stationary covariance function $c(t)$.

1.44 *Remark* A.44.

- The distribution of a stationary Gaussian process is invariant under translation in t.
- Examples are centered Gaussian processes with $C(s,t) = \delta(s-t)$, $\cos(s-t)$ or the exponential covariance function $e^{-|s-t|}$.

B Random Fields

2.1 **Definition B.1.** For $D \subset \mathbb{R}^d$, a real-valued random field $U = \{U(x) : x \in D\}$ is a family of real-valued r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2 **Example B.2** ($L^2(D)$-vaued r.v.)**.** *For $D \subset \mathbb{R}^d$, consider a $L^2(D)$-valued r.v. u with mean $\mu \in L^2(D)$ and covariance operator $\mathcal{C} : L^2(D) \to L^2(D)$ of the form*

$$\langle \mathcal{C}\varphi, \psi \rangle = \mathrm{Cov}[\langle w, \varphi \rangle, \langle v, \psi \rangle],$$

for $\varphi, \psi \in L^2(D)$ and general $L^2(D)$-valued r.v. w and v.

Then, $u(x)$ is a real-valued r.v. for each $x \in D$ for which $\mathbb{E}[u(x)]$ and $\mathrm{Cov}[u(x), u(y)]$ are well defined for $\mathrm{d}x$-a.e. $x, y \in D$.

For $\varphi, \psi \in L^2(D)$ it holds

$$\langle \mathcal{C}\varphi, \psi \rangle = \mathrm{Cov}[\langle \varphi, u \rangle, \langle \psi, u \rangle] = \mathbb{E}\left[\left(\int_D \varphi(x)(u(x) - \mu(x))\,\mathrm{d}x\right) \cdot \left(\int_d \psi(x)(u(y) - \mu(y))\,\mathrm{d}y\right)\right]$$
$$= \int_D \int_D \mathrm{Cov}[u(x), u(y)]\varphi(x)\,\mathrm{d}x\psi(y)\,\mathrm{d}y.$$

Thus,

$$(\mathcal{C}\psi)(x) = \int_D \mathrm{Cov}[u(x), u(y)]\varphi(y)\,\mathrm{d}y,$$

which shows that the kernel of $\mathcal{C}$ is $\mathrm{Cov}[u(x), u(y)]$. Hence any $L^2(D)$-valued r.v. u defines a second order random field with mean $u(x)$ and covariace function $C(x, y)$ equal to the kernel of $\mathcal{C}$.

2.3 **Theorem B.3** (Wiener-Khinchin)**.** *The following statements are equivalent:*

(i) There exists a stationary random field $u = \{u(x) : x \in \mathbb{R}^d\}$ wih stationary covariance function c and u is mean-square continuous.

(ii) The function $c : \mathbb{R}^d \to \mathbb{R}$ can be written as

$$c(x) = \int_{\mathbb{R}^d} e^{i\nu^T x}\,\mathrm{d}F(\nu)$$

for some finite measure F on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

*The measure F is called **spectral distribution**. If it exists, the density function f of F is called **spectral density** and*

$$c(x) = (2\pi)^{\frac{d}{2}}\tilde{\mathcal{F}}^{-1}[f](x) = \int_{\mathbb{R}^d} f(x)e^{i\lambda^T x}\,\mathrm{d}\lambda,$$

where $\tilde{\mathcal{F}}$ is the Fourier transform of F. Alternatively, given $c : \mathbb{R}^d \to \mathbb{R}$, the spectral density f is

$$f(\nu) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\nu^T x}c(x)\,\mathrm{d}x = (2\pi)^{-\frac{d}{2}}\tilde{\mathcal{F}}[c](\nu).$$

2.4 **Example B.4** (Gaussian Covariance)**.** *For symmetric positive definite $A \in \mathbb{R}^{d\times d}$ the stationary covariance function*

$$c(x) = e^{-x^T A x}, \quad x \in \mathbb{R}^d$$

has the Fourier transform

$$f(\nu) := \tilde{\mathcal{F}}[\nu] = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ix^T\nu}e^{-x^T A x}\,\mathrm{d}x = \frac{e^{-\frac{1}{4}\nu^T A^{-1}\nu}}{(2\pi)^{\frac{d}{2}}2^{\frac{d}{2}}\sqrt{\det A}}$$

Hence, f is the density of the measure F of the Gaussian distribution $\mathcal{N}(0, 2A)$ and thus Theorem B.3 implies that $c(x)$ is a valid stationary covarinace function.

2.5 **Definition B.5** (Isotropic Random Field)**.** A stationary random field is **isotropic** if the covariance $c(x)$ is invariant to rotations. Then, μ is constant and $c(x) = c^0(r)$ with $r := \|x\|$, with the **isotropic covariance function** $c^0 : \mathbb{R}^+ \to \mathbb{R}$.

2.6 **Example B.6.** *Let* $A = I_d$ *and* $c(x) = e^{-x^T A x}$. *Then,* $c^0(t) = e^{-r^2}$. c *is isotropic for any* $A = \sigma I_d$ *and* $\sigma > 0$.

2.7 *Remark* B.7 (KL of Random Fields). Analog to stochastic processes, we consider the $L^2(\Omega; L^2(D))$-convergent KL expansion

$$u(x,\omega) = \mu(x) + \sum_{j=1}^{\infty} \sqrt{\nu_j}\varphi_j(x)\xi_j(\omega)$$

with $\xi_j(\omega) := \frac{1}{\sqrt{\nu_j}} \langle u(x,\omega) - \mu(x), \varphi_j(x)\rangle_{L^2(D)}$ and $\{\nu_j, \varphi_j\}_{j\in\mathbb{N}}$ are eigenpairs of $\mathcal{C}$ with $\nu_1 \geq \nu_2 \geq \ldots \geq 0$, and

$$(\mathcal{C}\varphi)(x) = \int_D C(x,y)\varphi(y)\,\mathrm{d}y, \qquad x \in D,$$

where C is the covariance function of the random field u.

2.8 **Theorem B.8** (KL uniform convergence)**.** *If* $D \subset \mathbb{R}^d$ *is closed and bounded and* $C \in C(D \times D)$, *then* $\varphi_j \in C(D)$ *and*

$$\sup_{x,y\in D} |C(x,y) - C_J(x,y)| \leq \sup_{x\in D} \sum_{j=J+1}^{\infty} \nu_j\varphi_j(x)^2 \to 0, \qquad \text{as } J \to 0$$

with $C_J(x,y) = \sum_{j=1}^{J} \nu_j\varphi_j(x)\varphi_j(y)$ *and*

$$\sup_{x\in D} \mathbb{E}[(u(x) - u_J(x))^2] \to 0, \qquad \text{as } J \to \infty.$$

2.9 **Corollary B.9.** *If* $C(x,y) = c(x-y)$, *then*

$$\int_D \mathbb{V}[u(x)]\,\mathrm{d}x = c(0)|D| = \sum_{j=1}^{\infty} \nu_j \tag{12}$$ eqn:12

and

$$\int_D \mathbb{V}[u(x) - u_J(x)]\,\mathrm{d}x = c(0)|D| - \sum_{j=1}^{J} \nu_j.$$

Proof. We have

$$\mathbb{V}[u(x)] = C(x,x) = \sum_{j=1}^{\infty} \nu_j\varphi_j(x)^2.$$

Integrating over D gives

$$\int_D \mathbb{V}[u(x)]\,\mathrm{d}x = c(0)|D| = \sum_{j=1}^{\infty} \nu_j,$$

as $\|\varphi_j\|_{L^2(D)} = 1$. Moreover

$$\int_D \mathbb{V}[u(x) - u_J(x)]\,\mathrm{d}x = \mathbb{E}\left[\left\|\sum_{j=J+1}^{\infty} \sqrt{\nu_j}\varphi_j\xi_j\right\|_{L^2(D)}^2\right] = \sum_{j=J+1}^{\infty} \nu_j.$$

$\square$

2.10 *Remark* B.10. For numerical computations, the relative error is of importance

$$E_J := \frac{\int_D \mathbb{V}[u(x) - u_J(x)]\,\mathrm{d}x}{\int_D \mathbb{V}[u(x)]\,\mathrm{d}x} = \frac{c(0)|D| - \sum_{j=1}^{J} \nu_j}{c(0)|D|}.$$

- The decay rate of eigenvalues determines the decay of $E_J \to 0$.
- For a truncated KL, one has to solve the eigenvalue problem $\mathcal{C}\varphi = \nu\varphi$ for $\nu > 0$ and $\varphi \in L^2(D) \setminus \{0\}$.
- Analytical solutions are available only in a few special cases.

2.11 **Example B.11.** *(i) Exponential covariance in $d = 1$:*

$$C(x,y) = e^{-\frac{|x-y|}{l}} \qquad \text{on } D = [-a, a].$$

The associated eigenvalue problem is finding $\{\nu_i, \varphi_i\}$ such that

$$\int_{-a}^{a} e^{-\frac{|x-y|}{l}} \varphi(y)\,\mathrm{d}y = \nu\varphi(x), \qquad x \in D.$$

Differentiating two times leads to the ODE

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \omega^2\varphi = 0 \qquad \text{with } \omega^2 := \frac{2l^{-1} - l^{-2}\nu}{\nu}.$$

The solutions are

$$\varphi_i(x) = \begin{cases} A_i \cos(\omega_i x), & \text{if } i \text{ odd}, \\ B_i \sin(\omega_i x), & \text{if } i \text{ odd}. \end{cases}$$

and

$$\nu_i := \frac{2l^{-1}}{\omega_i^2 + l^{-2}}$$

with A_i, B_i such that $\|\varphi_i\|_{L^2(-a,a)} = 1$.

(ii) Separable exponential covariance for $d = 2$:

$$C(x,y) = \prod_{m=1}^{2} e^{-\frac{|x_m - y_m|}{l_m}} \qquad \text{on } D = [-a_1, a_1] \times [-a_2, a_2].$$

Then, we have $\varphi_j(x) = \varphi_i^1(x_1) \cdot \varphi_k^2(x_2)$ and $\nu_j = \nu_i^1 \nu_k^2$ with $\{\nu_i^1, \varphi_i^1\}$ and $\{\nu_k^2, \varphi_k^2\}$ eigenpairs of the 1-dimensional problems

$$\int_{-a_m}^{a_m} e^{-\frac{|x_m - y_m|}{l_m}} \varphi^m(y)\,\mathrm{d}y = \nu^m \varphi^m(x), \qquad x \in [-a_m, a_m], m \in \{1, 2\}.$$

In the following, for functions $f, h : \mathbb{R}^+ \to \mathbb{R}$, we use the notation "$f(s) \asymp h(s)$ as $s \to \infty$", read as f **is asymptotic to** h, if $\frac{f(s)}{h(s)} \to 1$ as $s \to \infty$.

2.12 **Theorem B.12** (Widom)**.** *Let $c^0(r)$ be an isotropic covariance function on $\mathbb{R}^d$ with spectral density function $f^0(s)$. Assume that $f'(s) \asymp bs^{-\varrho}$ as $s \to \infty$ for some $b, \varrho > 0$. Let D be a bounded domain in $\mathbb{R}^d$ and consider eigenvalues $\nu_1 \geq \nu_2 \geq \ldots \geq 0$ of the covariance operator $\mathcal{C}$. As $j \to \infty$*

$$\nu_j \asymp k(D, d, \varrho, b) \cdot j^{-\frac{\varrho}{d}} \quad \text{for} \quad k(D, d, \varrho, b) := (2\pi)^{d-\varrho} b(|D|V_d)^{\frac{\varrho}{d}}$$

2.13 **Example B.13.** *For the exponential covariance function* $c(x) = e^{-\frac{|x|}{l}}$ *and spectral density* $f(s) = \frac{l}{\pi(1+l^2s^2)}$ *it holds* $f(s) \asymp \frac{1}{\varphi l^2 s^2}$*, i.e.* $b = (\pi l)^{-1}$ *and* $\varrho = 2$.

2.14 **Example B.14** (Whittle-Matérn)**.** *We consider the covariance function*

$$c_q^0(r) = \frac{1}{2^{q_1\Gamma(q)}}\left(\frac{r}{l}\right)^q k_q\left(\frac{r}{l}\right)$$

with spectral density

$$f^0(s) = \frac{\Gamma\left(q+\frac{d}{2}\right)}{\Gamma(q)\pi^{\frac{d}{2}}} \cdot \frac{l^d}{(1+l^2s^2)^{q+\frac{d}{2}}}.$$

For $q = \frac{1}{2}$*,* $l = 1$ *is* $c^0_{\frac{1}{2}}(r) = e^{-\frac{r}{l}}$*, i.e. the exponential covariance.*

The parameter q *controls the regularity of the random field in* $\mathbb{R}^d$*. It is* n *times mean-squared differentiable if* $n < q$*.*

We see

$$f^0(s) \asymp bs^{-\varrho} \quad \textit{for} \quad b = l^{-2q} \cdot \frac{\Gamma\left(q+\frac{d}{2}\right)}{\Gamma(q)\pi^{\frac{d}{2}}}, \varrho = 2q + d.$$

Theorem B.12 yields

$$\nu_j \asymp (2\pi l)^{-2q}\frac{\Gamma\left(q+\frac{d}{2}\right)}{\Gamma(q)\pi^{\frac{d}{2}}}\left(\frac{|D|V_d}{j}\right)^{\frac{\varrho}{d}}.$$

2.15 **Theorem B.15** (E_J for Whittle-Matérn)**.** *Let* $D \subset \mathbb{R}^d$ *be closed, bounded and let* $u = \{u(x) : x \in D\}$ *be a random field with Whittle-Matérn covariance. Then*

$$E_J \asymp k_{q,d}\frac{1}{l^{2q}}\frac{1}{J^{\frac{2q}{d}}}$$

where

$$k_{q,d} := \frac{\Gamma\left(q+\frac{d}{2}\right)}{\Gamma(q)\varphi^{\frac{d}{2}}(2\pi)^{\frac{1}{q}}}|D|^{\frac{2q}{d}}V_d^{1+\frac{2q}{d}} \cdot \frac{d}{2q}$$

In the isotropic exponential case

$$E_J \asymp k_{\frac{1}{2},d}\frac{1}{lJ^{\frac{1}{d}}}.$$

2.16 *Remark* B.16 (Approximating Realisations). Suppose u is a gaussian field with mean function μ and covariance function C and known eigenpairs $\{\nu_j, \varphi_j\}$ of its covariance operator. Let $x_0, \dots, x_{N-1}$ be sample points and $\mathbf{u} = (u(x_0), \dots, u(x_{N-1}))^T$. Then $\mathbf{u} \sim \mathcal{N}(\mu, C)$ with $\mu = (\mu(x_0), \dots, \mu(x_{N-1}))^T$ nd C a matrix with entries $C_{i,j} = C(x_i, x_j)$.

An approximation of the random vector $\mathbf{u}$ can be calculated as $\mathbf{u}_J \sim \mathcal{N}(\mu, C_J)$ with C_J a matrix with entries $[C_J]_{i,j} = C_J(x_i, x_j) = \sum_{m=1}^J \nu_m\varphi_m(x_i)\varphi_m(x_j)$. Let $\varphi_j := (\varphi_j(x_0), \dots, \varphi_j(x_{N-1}))^T$, $j \in \{1, \dots, J\}$ and $\Phi_J = [\varphi_1, \dots, \varphi_J] \in \mathbb{R}^{N\times J}$ and $\Lambda_J = \mathrm{diag}(\nu_1, \dots, \nu_J) \in \mathbb{R}^{J\times J}$. Then

$$\mathbf{u}_J = \mu + \Phi_J\Lambda^{\frac{1}{2}}\xi$$

with $\xi \sim \mathcal{N}(0, I_J)$. Then the covariance matrix of $\mathbf{u}_J$ is $C_J = \Phi_J\Lambda\Phi_J^T$ and if it is close to C, then the distribution of $\mathbf{u}_J$ is close to the distribution of $\mathbf{u}$.

Recall : for $\varphi \in L^2(\mathbb{R}^N)$ we have

$$|\mathbb{E}[\varphi(\mathbf{u})] - \mathbb{E}[\varphi(\mathbf{u}_J)]| \leq \|\varphi\|_{L^2(\mathbb{R}^N)} \cdot \|C - C_J\|_F.$$

2.17 *Remark* B.17 (Approximating Eigenpairs). Note that discretizing $\mathcal{C}\varphi = \nu\varphi$ leads to large dense matrices.

Collocation: Let $x_1, \dots, x_p \in D$ and define the residuals

$$R_j(x) := \int_D C(x,y)\hat{\varphi}_j(y)\,\mathrm{d}y - \hat{\nu}_j\hat{\varphi}_j(x).$$

Approximating $\{\hat{\nu}_j, \hat{\varphi}_j\}$ requires $R_j(x_k) = 0$ for $k = 1, \dots, P$.

Assume we dispose of a quadtrature rule with points and weights $(x_k, q_k)_{k\in\{1,\dots,P\}}$, then

$$\sum_{i=1}^{p} q_i C(x_k, x_i)\varphi(x_i) \approx \int_D C(x_k, y)\varphi(y)\,\mathrm{d}y = \nu_j\hat{\varphi}_j(x_k)$$

Solving yields $\{\hat{\nu}_j, \hat{\varphi}_j\}$. In matrix notation this is $CQ\hat{\varphi}_j = \hat{\nu}_j\hat{\varphi}_j$ with $C \in \mathbb{R}^{P\times P}$, $Q \in \mathbb{R}^{P\times P}$ a diagonal weight matrix.

Conversion to a symmetric problem of the form $Sz_j = \hat{\nu}_j z_j$ is possible via $S := Q^{\frac{1}{2}}CQ^{\frac{1}{2}}$ and $z_j = Q^{\frac{1}{2}}\hat{\varphi}_j$.

Galerkin: Assume $V_P = \operatorname{span}\{\varphi_1, \dots, \varphi_P\}$ and determine $\hat{\varphi}_j \in V_P$ by solving

$$\int_D \int_D C(x,y)\hat{\varphi}_j(y)\,\mathrm{d}y\varphi_i(x)\,\mathrm{d}x = \hat{\nu}_j \int_D \hat{\varphi}_j\varphi_i(x)\,\mathrm{d}x, \qquad i \in \{1, dots, P\}.$$

or $\int R_j(x)\varphi_i(x)\,\mathrm{d}x = 0$ with

$$R_j(x) = \int_D C(x,y)\hat{\varphi}_j(y)\,\mathrm{d}y - \hat{\nu}_j\hat{\varphi}_j(x).$$

The choice $\varphi_j(x) = \delta(x - x_j)$ is the collocation approach.

Since $\hat{\varphi}_j(x) = \sum_{k=1}^{P} a_k^j\varphi_k(x)$ for some coefficients a_k^j

$$\sum_{j=1}^{P} a_k^j \int_D \int_D C(x,y)\varphi_i(y)\varphi_i(x)\,\mathrm{d}y\,\mathrm{d}x = \hat{\nu}_j \sum_{k=1}^{P} a_k^j \int_D \varphi_j(x)\varphi_i(x)\,\mathrm{d}x.$$

$\Rightarrow Ka_j = \hat{\nu}_j Ma_j$, which is a generalized eigenvalue problem with

$$K_{i,j} = \int_D \int_D C(x,y)\varphi_i(y)\varphi_j(x)\,\mathrm{d}y\,\mathrm{d}x \tag{B.1}$$

and $M_{i,j} = \int_D \varphi_i(x)\varphi_j(x)\,\mathrm{d}x$.